Wage Structure: Matching and Sorting

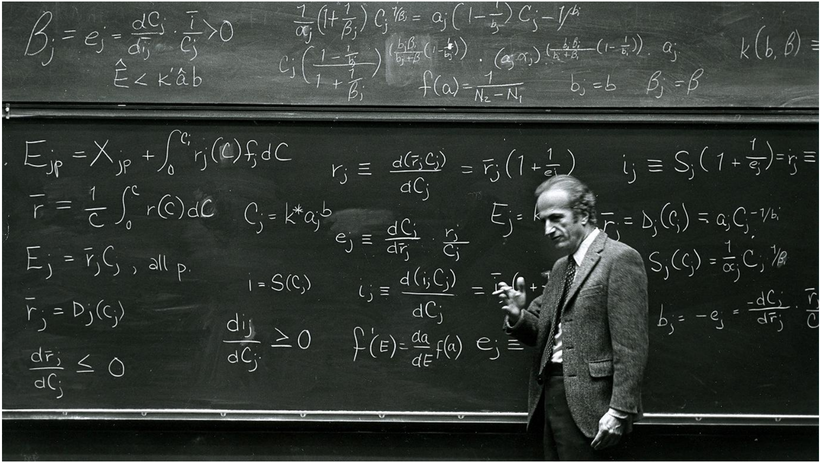
Xuanli Zhu
Keio University

Fall, 2023

Roadmap

1. Introduction
2. The theory of marriage
3. Applications in labor market

Gary Becker and ”Economic Imperialism”



”In recent years, economists have used economic theory more boldly to explain behavior outside the monetary market sector ... it is argued that marriage is no exception” —Becker (1973)

Marriage Market and Labor Market

- Marriage market
  - Individuals are fundamentally heterogeneous (by sex and by a list of characteristics or traits)
  - Individuals compete on the market with others in the same category
  - Individuals match or sort with their mates under mutual agreements
  - A match generates mutual benefit, i.e. a surplus, be it nonmonetary aspects (e.g. love) or economic benefits (e.g. shared consumption, gender specialization, risk sharing)
  - The division of the surplus between partners is endogenously determined through the matching process and market equilibrium

- Marriage market is a great analogy of the labor market in which firms match workers (Q: how?)

Matching vs Search

- Matching models: each woman (say) is assumed to have free access to the pool of all potential men
- Search models: each woman (say) sequentially and randomly meets one men, and decides whether to settle or continue searching
- We will focus on (frictionless) matching/sorting models and defer the discussion of search models to next week
- The choice of a specific model (frictionless matching vs search or other) should be driven by the empirical questions:
  - I.e. what are the main stylized features of the situation we want to investigate; how important frictions are likely to matter
  - A frictionless setting, though unrealistic, is acceptable if our focus is on board allocation patterns (e.g. matching by education)
  - A search model should probably be preferred if the focus is unemployment or mismatch where various frictions and matching processes matter

Roadmap

1. Introduction

2. The theory of marriage

3. Applications in labor market

Why marriage?

- Why ”two are better than one”? In other words, what are the joint surplus?

1. The sharing of public (nonrival) goods ‣ math example
   - E.g. both partners can use the same home and appliances, share the same information, and equally enjoy their children
2. The division of labor to exploit comparative advantage and increasing returns to scale ‣ math example
   - E.g. one partner works at home and the other works in the market
3. The coordination of home production
   - E.g. coordinating child care, which is a public good for the parents
4. Extending credit and coordination of investment activities
   - E.g. one partner works when the other is in school
5. Risk pooling
   - E.g. one partner works when the other is sick or unemployed

Transferable utility

- Within the family of frictionless matching frameworks, the key distinction relies on the nature of ”transfers”
  - I.e. whether a technology exists that would allow to transfer utility between agents participating in a matching process
  - It doesn’t need to be monetary transfers, in family match, it can be the allocation of time or expenditures for private or public goods
- When transfers are possible
  - Allow agents to ”bid” for their preferred mate by accepting to reduce their own gain from the match in order to increase the partner’s
  - Enables agents to negotiate, compromise, and ultimately exploit mutually beneficial solutions
  - The division of the surplus is endogenous and determined by equilibrium conditions on the marriage market
- Do marriage couples have utility transferable?
  - A spouse can reduce his or her private consumption to the partner’s benefit
  - Even if all public consumption, changing the composition of the bundle actually consumed typically results in utility transfers

Three cases

- (Prefectly) Transferable utility (TU) ▸ see math details
  - The transfer technology allows to transfer utility between agents at a constant one-to-one exchange rate
  - Individual utilities always add up to the total gain: $u(x) + v(y) = S(x, y)$
  - The Pareto frontier (the set of utility pairs that are just feasible given resource constraints) is a straight line with slope -1
- Imperfectly transferable utility (ITU)
  - More general version which allows the exchange rate between individual utilities is not constant and is typically endogenous to the economic environment (prices, incomes, etc.)
- Non-transferable utility (NTU)
  - Simply no technology enabling agents to decrease their utility to the benefit of a potential partner
  - E.g. kidney exchange, allocation of residents to hospitals, allocation of students to public schools

Equilibrium concept

- The equilibrium concept is a bit specific, namely, stability
  - On the other hand, it is probably the most intuitive and realistic equilibrium concept in economics
- Formally, we say that a matching is stable if
  1. There is no married person who would rather be single (often implicitly assumed)
  2. There are no two married (or unmarried) persons who prefer to form a new union
- The second condition is often referred to as divorce at will:
  - Whenever it is violated, the corresponding individuals will each divorce their current spouse (or abandon their current singlehood) to form a new union, implying that the initial matching was not stable
- In a frictionless world, a marriage structure that fails to satisfy (1) and (2) either will not form or will not survive

A motivational example

- Consider a matching market under NTU; The Gale-Shapley (1962) algorithm will discover the stable matching ‣ though need not be unique:

a

Payoffs	Y_1	Y_2	Y_3
X_1	2, 23	4, 24	**6, 25**
X_2	4, 16	**8, 18**	12, 20
X_3	**6, 9**	12, 12	18, 15

b

Sums	Y_1	Y_2	Y_3
X_1	**25**	28	31
X_2	20	**26**	32
X_3	15	24	**33**

Figure 1

Match payoffs and payoff sums. (*a*) The stable matching (*blue*) for the nontransferable utility match payoffs. (*b*) Once monetary or utility transfers are allowed (the transferable utility case), the corresponding payoff sums are the relevant benchmark, and the stable matching (*red*) switches persons.

- Next observe how these matches are inefficient; The above unique NTU matching is thus unstable once transfers are allowed *(and, as they say, money changes everything!)*
  - Note that in (a) the stable matching is assortative (better X matches better Y); In general, assortative matching is more easily to achieve under NTU than TU
- With TU, a stable assignment maximizes total output over all possible assignments and the payoffs are now endogenous and determined in equilibrium

The basic model

▸ generalize to finite agents ▸ generalize to continuous agents

- Consider three men with feature $x \in \mathcal{X} = \{1, 2, 3\}$ and three women with feature $y \in \mathcal{Y} = \{1, 2, 3\}$
- Match surplus function: $S(x, y) = x \cdot y = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$
- Under TU, we have $u(x) + v(y) = S(x, y)$
- An allocation μ indicates which man is matched with which woman: $y = \mu(x)$, e.g. $1 = \mu(3), 2 = \mu(2), 3 = \mu(1)$
- An equilibrium with outcome (u, v, μ) is stable if there is no blocking pair, i.e., $u(x) + v(y) \geq S(x, y), \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$
- E.g. if $\{2, 2\}$ is a stable match in the equilibrium, we should have
  $u_2 + v_1 \geq S(2, 1); u_2 + v_2 = S(2, 2); u_2 + v_3 \geq S(2, 3);$
  $u_1 + v_2 \geq S(1, 2);$ $\quad u_3 + v_2 \geq S(3, 2)$

The basic model (cont.)

- This implies a system of 9 inequalities, and along the equilibrium allocation, 3 hold with equality: $u(x) + v(\mu(x)) = S(x, \mu(x))$
- The solution of stable equilibrium satisfies $\mu(1,2,3) = (1,2,3)$ and $1 \leq u_2 - u_1 \leq 2, 2 \leq u_3 - u_1 \leq 6, 2 \leq u_3 - u_2 \leq 3$
  - These conditions come from the stability in equilibrium: e.g. $v_1 = 1 - u_1 \geq 2 - u_2$, $v_2 = 4 - u_2 \geq 2 - u_1$, ...
  - The sum of all match surplus $S(1,1) + S(2,2) + S(3,3)$ is the maximum among all possible allocations (easy to see in a 2x2 case), which means we can obtain the stable allocation by solving an assignment problem $\max_\mu \sum_{x \in \mathcal{X}} S(x, \mu(x))$
- Thus a unique allocation that exhibits positive assortative matching (PAM), but there are multiple (u, v) that are consistent with this stable allocation
  - One particular solution is $u = (0.5, 2, 4.5)$, $v = (0.5, 2, 4.5)$
  - This indeterminacy disappears when continuous distributions of x, y
- Replace men and women with workers and firms, S with F (production function), and u, v with w, π (wage and profit), we immediately get a model of labor market matching

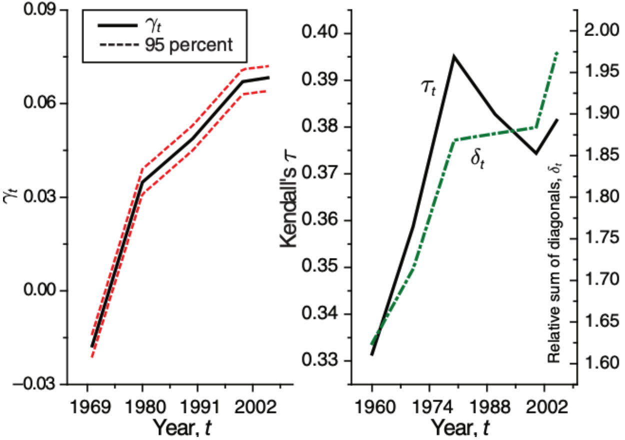
Assortative mating

▸ see the derivation under decentralized competitive equilibrium

- Under TU, the condition for PAM is that the joint surplus (or output) function $S(x, y)$ is supermodular: if $x' > x$ and $y' > y$, $S(x', y') + S(x, y) \geq S(x', y) + S(x, y')$
  - NAM if S is submodular, i.e. the inequality is reversed
  - This definition captures the idea of complementarity (substitution)
  - When S is twice differentiable, then S is super-(sub-)modular if the second cross-derivative S_{xy} is always positive(negative)
- The intuition follows from the observation that a stable assignment must maximize the aggregate marital surplus over all possible assignments ▸ graphical interpretation
  - Supermodular means the sum of its value at the extremes exceeds that at the intermediates, i.e. a notion of convexity on a multidimensional domain
- An alternative interpretation can be thought of in terms of increasing differences: $S(x', y') - S(x', y) \geq S(x, y') - S(x, y)$
  - I.e. the contribution to marital output of a given increase in the female attribute rises with the level of male trait (and symmetrically)

Empirical finding: rise in assortative mating ▸ more details

- ▷ Greenwood et al. (2014) consider a regression between a wife's educational level and her husband's:
$$\mathrm{EDU}^{w}_{my} = \alpha + \beta \mathrm{EDU}^{h}_{my} + \sum_{t \in \mathcal{T}} \gamma_t \mathrm{EDU}^{h}_{my} \mathrm{YEAR}_{ty} + \sum_{t \in \mathcal{T}} \theta_t \mathrm{YEAR}_{ty}$$



- ▷ (Q: why?)

Some other interesting empirical findings

- Fernandez et al. (2005): Greater inequality may tend to make matches between different classes of individuals less likely, as the cost of ”marrying down” increases
- Chiappori et al. (2012): For women, an additional year of education may compensate up to two BMI units, and men may compensate a 1.3-unit increase in BMI with a 1 percent increase in wages. Interestingly, male physical attractiveness matters as well.
- Ciscato et al. (2020): As concerns age and ethnicity, different-sex couples exhibit a higher degree of assortativeness than same-sex ones; However, sorting on education is stronger among lesbians with respect to different-sex couples

Roadmap

1. Introduction

2. The theory of marriage

3. Applications in labor market

Assignment models

- Assignment problem describes a general class of settings in which a group of unique factors are allocated to one another: men to women, workers to firms, doctors to patients, athletes to teams, ...
- Assignment models were introduced by Tinbergen in 1950s, and popularized (made less obscure) by Sattinger (1993)
- Sattinger considers three ways to generate assignment pattern: comparative advantage, scale of operations, and preference
- Consider a case of production:

	Job1	Job2
Worker1	\$36	\$24
Worker2	\$20	\$10

- Under Roy's comparative advantage (and wage is a fixed proportion of production), both worker types will select job1 (Q: what if wages differ across jobs?) see how Ricardo's C.A. is related to supermordularity
- However, under the matching framework with limited positions of job1 (e.g. manager, artist), we know the stable matching is diagonal

O-ring theory: setting

- Assume a firm using a production process consisting of n tasks
  - Assume each task can be assigned to only one worker and n is technologically fixed (can be relaxed to be endogenous)
  - A distinctive feature of assignment models: productive resources are embedded in indivisible units, and these units must be combined in fixed numbers to produce output

- A continuum of workers with exogenous distribution of skill/quality, $\phi(q)$; A worker's skill $q \in (0, 1]$ is the percentage of maximum value the product retains

- Firm production: $y = \left(\Pi_{i=1}^{n} q_i\right) nB$, where nB is perfect output
  - Note that there is no perfect substitution between workers of different q under fixed positions/tasks
  - Complementary as cross derivative $\frac{d^2 y}{dq_i d\left(\Pi_{j \neq i} q_j\right)} = nB > 0$

- A competitive equilibrium is defined as an assignment of workers to firms, and a wage profile, $w(q)$, such that firms maximize profits and the market clears for workers of all skill levels

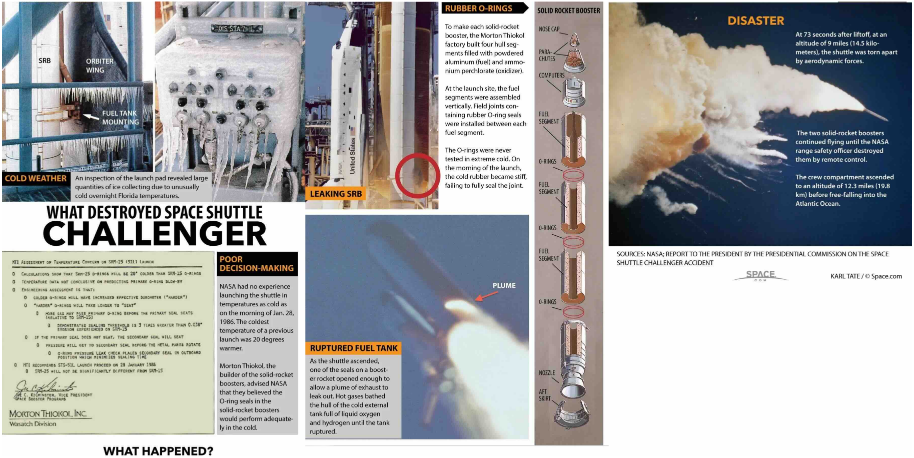
O-ring theory: firm problem and solution

- Firm problem: $\max_{\{q_i\}} \left(\Pi_{i=1}^{n} q_i\right) nB - \sum_{i=1}^{n} w\left(q_i\right)$
- Under PAM, the equilibria can be restricted to those allocations of same q workers in any single firm
  - Intuitively, firms with high q workers in the first $n-1$ tasks bid the highest value on having high-skill workers in the nth task
- Thus rewrite the firm problem: $\max_q q^n nB - nw\left(q\right)$
- FOC on q: $\frac{dw}{dq} = q^{n-1} nB$
- Integrating the FOC generates the equilibrium wage schedules: $w(q) = q^n B + c$
  - Constant of integration, c, represents the wage of a worker of lowest skill (0)
  - Wage bill: $nw(q) = y + nc$
  - Zero profit condition implies that $c = 0$
- Equilibrium holds when firms demand the number of workers of each skill available in the population

Implications on labor markets

1. Provides a mechanism through which small differences in worker skill create large differences in productivity and wages
   - ▷ Note equilibrium wages are not only increasing in q but actually homogenous of degree $n/(1-\alpha)$ in q (IRS)
2. Account for why different firms hire different qualities of workers
   - ▷ McDonald's does not hire famous chefs, and Maxim's does not hire teenage waiters
3. Account for industrial-level or firm-level wage premium
   - ▷ While pressures for intrafirm equity and industry rents have been suggested, O-ring production functions suggests the highest q secretaries will work with the highest q lawyers and bankers
4. Account for why firms only offer jobs to some workers rather than paying all workers their estimated marginal product
   - ▷ Under O-ring, the firm needs workers whose skill matches that of its current workers, hence be willing to interview many for a position
5. Account for the right-skewed income distribution
   - ▷ If q is distributed symmetrically, y and w will be skewed to the right

Why is it called "O-ring theory"?



"Many production processes consist of a series of tasks, mistakes in any of which can dramatically reduce the product's value. The space shuttle Challenger had thousands of components: it exploded because it was launched at a temperature that caused one of those components, the O-rings, to malfunction."—Kremer (1993)

Segregation by skill

- In the O-ring model, we have segregation of workers by skills across firms/workplaces
- This feature does not exist in the canonical model of skill premiums, where we also have different skills to be imperfect substitutes and different tasks within a firm to be complementary
- The key difference is that in the SBTC model, skills and tasks matched in a fixed way, and thus the assignment is exogenous
  - Another way to think is that a low-skill worker has productivity 0 in a high-skill task, and vice versa (like ”men” cannot be ”wives”)
  - Within a certain task, the assignment is undetermined due to perfect substitutability across same type workers
- Therefore, it is the combination of task complementarity and free allocation of skills into tasks that generates sorting/segregation, though here in another rather restrictive way
  - The O-Ring setting assumes that one worker is equally efficient in all tasks, i.e. the human capital is uni-dimensional and fully general
  - This assumption thus yields (not-very-realistic) complete segregation

Theory of CEO pays: setting

- Terviö (2008); Gabaix and Landier (2008): Why has CEO pay increased and varied so much?
  - Sixfold increase of U.S. CEO pay between 1980 and 2003
  - Large firms pay their CEOs significantly more than small firms do
  - Could differences in talent be able to explain such pay levels?
- Assume a unit mass of individual managers and firms are matched one to one
- Individuals are ordered by their ability: $a[i]$ is the ability of an i quantile(rank) individual and $a'[i] > 0$, with distribution function $F_a(a) = i$; Similarly firms are ordered by their size: $b[i]$
- The production function $Y(a, b)$ features complementarity (a positive cross-partial), and thus efficiency requires PAM
- Equilibrium matching is thus very simple, as is the equilibrium output; It is the division of output into factor incomes (wages and profits) that requires further analysis

Theory of CEO pays: equilibrium conditions

- In competitive equilibrium, the profiles of factor incomes must support the efficient matching of individuals and firms
- Two types of conditions must hold in competitive equilibrium
- 1 Sorting constraints:
  $Y(a[i], b[i]) - w[i] \geq Y(a[j], b[i]) - w[j] \forall i, j \in [0, 1] \mathrm{SC}(i, j)$
  - If there were n workers and n firms, there are $2n!$ sorting constraints
  - However, most constraints are redundant since for $i \geq j \geq k$, $SC(i, j) + SC(j, k)$ implies $SC(i, k)$
- 2 Participation (Incentive compatibility) constraints:
  $$\begin{array}{lll} Y(a[i], b[i]) - w[i] \geq \pi^0 & \forall i, \in [0, 1] & \mathrm{PC} - b[i] \\ w[i] \geq w^0 & \forall i, \in [0, 1] & \mathrm{PC} - a[i] \end{array}$$
  - Assume outside options (w^0, π^0) same for all units and lowest active pair $(i = 0)$ breaks even: $Y(a[0], b[0]) = \pi^0 + w^0$ more details
- Thus binding constraints are
  - Marginal sorting constraints that keep firms from wanting to hire the next best individual
  - Participation constraints of the lowest types

Theory of CEO pays: equilibrium wages and profits

- Regrouping the sorting constraint $\mathrm{SC}(i, i-\varepsilon)$: $\frac{Y(a[i],b[i])-Y(a[i-\varepsilon],b[i])}{\varepsilon} \geq \frac{w[i]-w[i-\varepsilon]}{\varepsilon}$, which becomes an equality as $\varepsilon \to 0$ and yields the slope of wage profile: $w'[i] = Y_a(a[i], b[i])a'[i]$

- Integrating to get wage profile: $w[i] = w^0 + \int_0^i Y_a(a[j], b[j])a'[j]dj$

- Analogously: $$\pi'[i] = Y_b(a[i], b[i])b'[i]$$ $$\pi[i] = \pi^0 + \int_0^i Y_b(a[j], b[j])b'[j]dj$$

- All inframarginal pairs produce a surplus over the sum of their outside opportunities, and the division of this surplus depends on the distributions of factor quality
  - At any given point in the profile, the increase in surplus is shared between the factors in proportion to their contributions to the increase at that quantile
  - Due to the continuity assumptions, the factor owners do not earn rents over their next best opportunity within the industry
  - Factor owners (i) are affected by changes in the quality of only those below them in the rankings $(0, i-\varepsilon)$

Comparative Statics: Change in the Shape of a Distribution

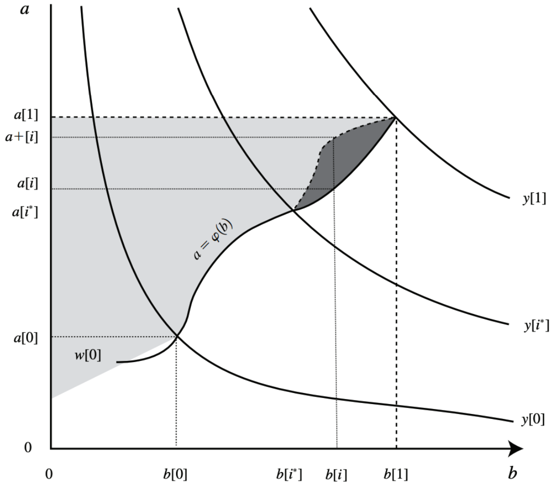


FIGURE 1. COMPARATIVE STATICS IN THE MULTIPLICATIVE CASE

Notes: The increasing curve covers both the active matches $\{b, \varphi(b)\}$ above $b[0]$ and potential but inactive matches below $b[0]$. The three decreasing curves are the isoquants for levels of output $y[0]$, $y[i^*]$, and $y[1]$. The entire shaded region is the equilibrium wage of the highest ability type, $a[1]$. The dark shaded region is the decrease in wage and increase in profits for the highest types if the matching curve between quantiles i^* and 1 were to shift up to the dashed line.

Reference I

Becker, G. S. (1973). A theory of marriage: Part i. *Journal of Political economy 81*(4), 813–846.

Chiappori, P.-A., S. Oreffice, and C. Quintana-Domeque (2012). Fatter attraction: anthropometric and socioeconomic matching on the marriage market. *Journal of Political Economy 120*(4), 659–695.

Ciscato, E., A. Galichon, and M. Goussé (2020). Like attract like? a structural comparison of homogamy across same-sex and different-sex households. *Journal of Political Economy 128*(2), 740–781.

Fernandez, R., N. Guner, and J. Knowles (2005). Love and money: A theoretical and empirical analysis of household sorting and inequality. *The Quarterly Journal of Economics 120*(1), 273–344.

Gabaix, X. and A. Landier (2008). Why has ceo pay increased so much? *The quarterly journal of economics 123*(1), 49–100.

Greenwood, J., N. Guner, G. Kocharkov, and C. Santos (2014). Marry your like: Assortative mating and income inequality. *American Economic Review 104*(5), 348–353.

Kremer, M. (1993). The o-ring theory of economic development. *The quarterly journal of economics 108*(3), 551–575.

Reference II

Kremer, M. and E. Maskin (1996). Wage inequality and segregation by skill.

Lucas, R. E. (1978). On the size distribution of business firms. *The Bell Journal of Economics*, 508–523.

Rosen, S. (1981). The economics of superstars. *The American economic review 71*(5), 845–858.

Rosen, S. (1982). Authority, control, and the distribution of earnings. *The Bell Journal of Economics*, 311–323.

Sattinger, M. (1993). Assignment models of the distribution of earnings. *Journal of economic literature 31*(2), 831–880.

Terviö, M. (2008). The difference that ceos make: An assignment model approach. *American Economic Review 98*(3), 642–668.

Appendix

Why marriage surplus: public consumption

- Assume utility functions be given by $u^s(Q, q^s) = q^s Q$ for $s = m, f$
  - q^s denote the consumption of a private good by person s
  - Q denotes the consumption of a public good which can be potentially shared with others if they form a family

- If the two agents live apart, then each individual s solves $\max_{Q,q^s} q^s Q$ s.t. $Q + q^s = y^s$
  - Optimal solution: $\hat{Q}^s = \hat{q}^s = y^s/2$ and thus $u^s = (y^s/2)^2$

- If they live together, they can pool their income, optimize their consumption on public goods subject to their joint budget constraint $Q + q^m + q^f = y^m + y^f \equiv Y$
  - Optimal solution: $\hat{Q} = Y/2$, $q^m + q^f = Y/2$ and thus $u^s = (Y/2)\, q^s \Rightarrow u^m + u^f = (Y/2)^2$

- If $y^f = 1$ and $y^m = 3$, alone, partner f obtains $u^f = \frac{1}{4}$ and partner m obtains $u^m = \frac{9}{4}$; together, $u^m + u^f = 4$ and clearly we can find the allocation of (q^m, q^f) such that both have higher utility than staying alone, i.e. it's beneficial to form family

Why marriage surplus: specialization and IRS

- Assume that agents only derive utility from a single nonmarket good z with household production function is $z = xt$
  - x denotes expenditure on market good inputs $x = w^s(1-t)$
  - $t \in [0,1]$ denotes time spent on home production
- An agent living alone maximize z and sets $t_s = \frac{1}{2}, z_s = \frac{w^s}{4}$
- If the couple lives together, we assume that f and m are perfect substitutes in home production $z = x\left(t^f + t^m\right)$, and that they agree to maximize the total output (before divide it) $z = \left[w^f\left(1-t^f\right) + w^m\left(1-t^m\right)\right]\left(t^f + t^m\right)$
  - Thus we are assuming z is a private good and there is no joint benefits from public consumption
- If the couple set the time allocation same as singles, their total output will be $\left(w^f + w^m\right)/2$, which is larger than $\left(\frac{w^f}{4} + \frac{w^m}{4}\right)$ due to increasing returns
- Suppose $w^m > w^f$, and set $t^m = 0$ and $t^f = 1$, the couple's total output will be w^a, which is greater than the output $\left(w^a + w^b\right)/2$ due to specialization according to comparative advantage

When do we have TU under Family Model?

- Recall what we learned about the family framework of labor supply; In which case do we have TU?
- TU holds if we have egotistic preferences and each felicity function can be put into a form that is similar to the Gorman polar form:
  $u^a(\mathbf{Q}, \mathbf{q}^a) = f^a(q_2^a, \ldots, q_n^a, \mathbf{Q}) + G(\mathbf{Q}) q_1^a$
  $u^b\left(\mathbf{Q}, \mathbf{q}^b\right) = f^b\left(q_2^b, \ldots, q_n^b, \mathbf{Q}\right) + G(\mathbf{Q}) q_1^b$ with $G(\mathbf{Q}) > 0\ \forall \mathbf{Q}$
  - $\mathbf{q}$ is a n-vector of private goods $\{q_1, \ldots q_n\}$, and each private good bought is divided between the two partners so that $\mathbf{q}^a + \mathbf{q}^b = \mathbf{q}$
  - $\mathbf{Q}$ is a N-vector of public goods $\{Q_1, \ldots Q_N\}$
  - An allocation is an $N + 2n$-vector $(\mathbf{Q}, \mathbf{q}^a, \mathbf{q}^b)$
  - f^a, f^b allows each married person has her or his own preferences over the allocation of family resources
  - The key is that G function is identical for both members attached on the consumption of q_1
- In words, TU assumption implies utility can be transferred between them, using (at least) commodity 1, at a fixed rate of exchange, since its marginal utility is always the same for both members

Who propose matters under NTU

- A case where preferences diverge among men and women:

	F_1	F_2	F_3
M_1	$3, 2$	$2, 6$	$1, 1$
M_2	$4, 3$	$7, 2$	$2, 4$
M_3	$1, 1$	$2, 1$	$0, 0$

- If men propose:

	F_1	F_2	F_3
M_1	$\mathbf{3}, \mathbf{2}$	$2, 6$	$1, 1$
M_2	$4, 3$	$\mathbf{7}, \mathbf{2}$	$2, 4$
M_3	$1, 1$	$2, 1$	$\mathbf{0}, \mathbf{0}$

- If women propose:

	F_1	F_2	F_3
M_1	$3, 2$	$\mathbf{2}, \mathbf{6}$	$1, 1$
M_2	$\mathbf{4}, \mathbf{3}$	$7, 2$	$2, 4$
M_3	$1, 1$	$2, 1$	$\mathbf{0}, \mathbf{0}$

- Bot matchings are stable $\Rightarrow$ Social norms of courting can have a large impact on matching patterns (esp. in the case of dating)!

Generalize matching with a finite number of agents

- Consider the general assignment problem with M males and N females

- Denote $z_{ij} = \zeta_{ij} - \zeta_{i0} - \zeta_{0j}$ as the marital joint surplus
  - ζ_{ij} is the total output of a marriage between male i and female j
  - ζ_{i0} (ζ_{0j}) is the utility that person i (j) receives as single

- Define assignment indicators a_{ij} such that $a_{ij} = 1$ iff i is married to j and $a_{ij} = 0$ otherwise; also $a_{i0} = 1$ ($a_{0j} = 1$) if i (j) is single

- The stable assignment is a solution to an integer linear programming problem faced by a social planner: $\max_{a_{ij}} \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij}\zeta_{ij}$ s.t. $a_{ij} \geq 0$ and $\sum_{j=0}^{N} a_{ij} = 1, \sum_{i=0}^{M} a_{ij} = 1$

- Since $a_{0j} = 1 - \sum_{i=1}^{M} a_{ij}$ and $a_{i0} = 1 - \sum_{j=1}^{N} a_{ij}$, the program can be rewritten as $\max_{a_{ij}} \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} \left(\zeta_{ij} - \zeta_{i0} - \zeta_{0j}\right) + C = \max_{a_{ij}} \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} z_{ij} + C$ s.t. $\sum_{j=1}^{N} a_{ij} \leq 1, \sum_{i=1}^{M} a_{ij} \leq 1$
  - where $C = \sum_{i=1}^{M} \zeta_{i0} + \sum_{j=1}^{N} \zeta_{0j}$ is the aggregate utility of singles

Linear programming problem and dual problem

- A standard tool of linear programming is duality theory: if we have an original maximization problem (the primal) $\max_{\mathbf{x}} z = \mathbf{c}^T\mathbf{x}$ s.t. $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ then the dual problems is $\min_{\mathbf{y}} w = \mathbf{b}^T\mathbf{y}$ s.t. $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq 0$, where $\mathbf{y}$ is the vector of dual variables associated with the constraints of the primal problem

- Applying to our surplus maximization problem, we can define a dual program: $\min_{u_i, v_j} \left(\sum_{i=1}^{M} u_i + \sum_{j=1}^{N} v_j \right)$ s.t. $u_i \geq 0$, $v_j \geq 0$, and $u_i + v_j \geq z_{ij}$
  - Optimal values of u_i and v_j can be interpreted as shadow prices (Lagrange multipliers) of the constraints in the original problem, which describes the social cost of moving a particular man (woman) away from the pool of singles
  - If the sum of social costs $u_i + v_j$ exceed the social gain z_{ij}, the particular marriage would not form; If a marriage is formed, then $u_i + v_j = z_{ij}$

- The key observation here is that $u_i + v_j \geq z_{ij}$ are nothing else than the stability conditions, and shadow prices u_i and v_j are simply the share of the surplus received at the stable matching

Decentralization of the stable matching

- These previous results have a nice interpretation in terms of decentralization of the stable matching
- A stable assignment can be supported (implemented) by a reservation utility vector whereby male i enters the market with a reservation utility u_i and is selected by the woman who gains the highest surplus $z_{ij} - u_i$ from marrying him
  - I.e. each male i faces the problem $\max_j z_{ij} - u_i$ and takes u_i as given
  - Similarly, woman j enters with a reservation utility v_j and is selected by the man who has the highest gain $z_{ij} - v_j$ from marrying her.
- In equilibrium, each agent receives a share in marital surplus that equals his or her reservation utility
  - In a sense, u_i and v_j can be thought of as the ”price” that must be paid to marry Mr. i or Mrs. j; each agent maximizes his or her welfare taking as given this ”price” vector

TU: Notations

- Two compact sets $\bar{X} \subset \mathbb{R}^n$ and $\bar{Y} \subset \mathbb{R}^m$ represent the space of female and male characteristics

- The vectors of characteristics fully describe the agents; i.e., for any $\mathbf{x} \in \bar{X}$, two women with the same $\mathbf{x}$ are perfect substitutes as far as matching is concerned (and similarly for men)

- Two spaces are endowed with measures F and G respectively

- In order to capture the case of persons remaining single within this framework, consider the spaces $X := \bar{X} \cup \{\emptyset_X\}$, $Y := \bar{Y} \cup \{\emptyset_Y\}$, where the point $\emptyset_X$ ($\emptyset_Y$) is dummy partner for any unmatched and endowed with a mass measure equal to the total measure of $\bar{Y}$ ($\bar{X}$)

- Defines a measure b on $X \times Y$; intuitively, one can think of $b(x, y)$ as the probability that x is matched to y

TU: matching and equilibrium

- A matching under TU is defined by
  - A measure b satisfying $\int_{\mathbf{y}\in Y} db(\mathbf{x},\mathbf{y}) = F(\mathbf{x})$, $\int_{\mathbf{x}\in X} db(\mathbf{x},\mathbf{y}) = G(\mathbf{y})$
  - Two individual utility functions $u(\mathbf{x})$ and $v(\mathbf{y})$ such that $b(\mathbf{x},\mathbf{y}) > 0 \Rightarrow u(\mathbf{x}) + v(\mathbf{y}) = S(\mathbf{x},\mathbf{y})$, i.e. matched people share the resulting surplus (Singles' utility is normalized to zero)

- The equilibrium condition is stability. which requires $u(\mathbf{x}) + v(\mathbf{y}) \geq S(\mathbf{x},\mathbf{y})\ \forall(\mathbf{x},\mathbf{y}) \in X \times Y$

- An equivalent statement: If a matching is stable, the corresponding functions u and v, are such that
$$u(\mathbf{x}) = \max_{z\in Y}\{S(\mathbf{x},\mathbf{z}) - v(\mathbf{z})\},$$
$$v(\mathbf{y}) = \max_{z\in X}\{S(\mathbf{z},\mathbf{y}) - u(\mathbf{z})\},$$
the maximum being reached in each case for potential spouses to whom the individual is matched with positive probability

- A natural interpretation: $v(\mathbf{y})$ is the price (in utility terms) that Mrs. $\mathbf{x}$ would have to pay should she choose to marry Mr. y, and then she would keep what is left of the surplus, namely $S(\mathbf{x},\mathbf{y}) - v(\mathbf{y})$

TU: surplus maximization

- Find a measure b on $X \times Y$ that maximizes the integral $\mathcal{S} = \int_{X \times Y} S(\mathbf{x}, \mathbf{y}) db(\mathbf{x}, \mathbf{y})$
  - In economics, a straightforward interpretation: a benevolent dictator who can match people at will and is trying to maximize total welfare

- This problem is linear in b; one can thus apply the results of duality theory; The dual problem is: Find two functions u and v that minimize the sum $\tilde{\mathcal{S}} = \int_X u(\mathbf{x}) dF(\mathbf{x}) + \int_Y v(\mathbf{y}) dG(\mathbf{y})$ s.t. $u(\mathbf{x}) + v(\mathbf{y}) \geq S(\mathbf{x}, \mathbf{y})\ \forall (\mathbf{x}, \mathbf{y}) \in X \times Y$ (the stability constraints!)

- The main result is that under mild conditions, if b [resp. (u, v)] is a solution to the primal (dual) problem, then (b, u, v) define a matching, and this matching is moreover stable
  - Thus finding a stable matching boils down to the resolution of a linear maximization problem, which is more tractable (particularly when numerical simulations are involved, since it boils down to linear programming) than finding (b, u, v) satisfying stable matching

TU: supermodularity

- In the one-dimensional case $m = n = 1$, the supermodularity of the surplus is defined: For all x, x', y, y' such that $x \leq x'$ and $y \leq y'$, we have $S(x, y) + S(x', y') \geq S(x, y') + S(x', y)$, and when S is twice continuously differentiable, this is equivalent to a standard Spence-Mirrlees condition, $\frac{\partial^2 S}{\partial x \partial y}(x, y) \geq 0 \quad \forall x, y$
- One can readily see that when S is supermodular, then the only stable matching must be assortative: for any two matched couples (x, y) and (x', y') such that $x \leq x'$, we must have that $y \leq y'$
- Then matching patterns follow, among married couples, a simple rule: x is matched to y iff the total mass of matched women above x equals the total mass of matched men above y
- Formally, the matching is pure, and (assuming atomless distributions and an identical mass of men & women) we obtain $1 - F(x) = 1 - G(y) \Rightarrow y = \phi(x) = G^{-1} \circ F(x)$
- If the opposite inequality, then the surplus function is submodular, and the stable matching is negative assortative

TU: A simple family model

- Individual preferences: $u_i\left(C_i, Q\right) = C_i Q$, where $i = 1, 2$
  - C_i is the private consumption
  - Q is a public good, domestically produced according to $Q = \left(t_1 t_2\right)^{\alpha}$

- Household, after matched, maximizes the joint utility $S \equiv u_1 + u_2 = CQ$, where $C \equiv \left(C_1 + C_2\right)$
  - Thus the model satisfies the TU conditions: any (interior) efficient allocation must maximize the sum of utilities

- Household budget constraint: $C + w_1 t_1 + w_2 t_2 = w_1 + w_2$, where $w_i = W H_i$ (agents differ by their human capital H_i)

- FOCs: $$\begin{aligned} w_1 Q^* &= C^* \alpha Q^* / t_1 \\ w_2 Q^* &= C^* \alpha Q^* / t_2 \end{aligned}$$

- $\Rightarrow C = \frac{w_1 + w_2}{1 + 2\alpha}$ and $w_1 t_1 = w_2 t_2 = \frac{\alpha\left(w_1 + w_2\right)}{1 + 2\alpha}$

- $\Rightarrow S\left(H_1, H_2\right) = \frac{\alpha^{2\alpha}}{\left(1 + 2\alpha\right)^{1 + 2\alpha}} W \left(H_1 + H_2\right)^{1 + 2\alpha} H_1^{-\alpha} H_2^{-\alpha}$

TU: A simple family model (cont.)

- $\frac{\partial^2 S(H_1,H_2)}{\partial H_1 \partial H_2} = -\frac{\alpha^{2\alpha+1} W (H_1+H_2)^{2\alpha-1}}{(2\alpha+1)^{2\alpha+1} H_1^{\alpha+1} H_2^{\alpha+1}} \left(H_1^2 + H_2^2 + \alpha (H_1 - H_2)^2 \right) < 0$
- Thus S is submodular, and the stable matching is negative assortative
  - I.e. high-HC men marry low-HC women (and vice versa)
  - Recall $w_1 t_1 = w_2 t_2$, thus low-wage people devote much time to domestic production and high-wage spouses concentrate on market work, reflecting the idea of marriage gain from specialization
  - The key (and tricky) assumption that gives this result is that efficiency in home production is irrelevant to efficiency in market
- Now change the household production function to $Q = (H_1 t_1)^{\alpha/2} (H_2 t_2)^{\alpha/2}$, and the surplus becomes $S(H_1, H_2) = W \frac{\alpha^{2\alpha} T^{1+2\alpha}}{(1+2\alpha)^{1+2\alpha}} (H_1 + H_2)^{1+2\alpha}$
  - Now $\partial^2 S / \partial H_1 \partial H_2 > 0$ and the surplus is supermodular, generating positive assortative matching

Competitive equilibrium: continuum of agents

- Let male type x and female type y be distributed according to $F(x)$ and $G(y)$
- $\mu(x) = y$, $u(x)$, $v(y)$ and $S(x, y)$ as before
- Now analyze this problem as a competitive equilibrium, where women y choose their optimal men x to match with, taking as given a market surplus schedule $u(x)$
  - It can be symmetrically analyzed as men x choose women y
  - $u(x)$ and $v(y)$ can be thought of as the market "price" that must be paid to marry Mr. x or Mrs. y
  - The assignment problem, the stable matching, and the competitive equilibrium all coincides; Equilibrium allocation is thus optimal: it maximizes total sum of the surplus
- Model is closed with a market clearing condition, basically ensuring that the matching μ is measure preserving: the measure of x matched is equal to the measure of y

Competitive equilibrium: solution

- Given a market schedule $u(x)$, each women y maximizes surplus: $v(y) = \max_{\tilde{x}} S(\tilde{x}, y) - u(\tilde{x})$ with FOC $S_x(x, y) - \frac{\partial u(x)}{\partial x} = 0$
- Obtain the equilibrium schedule by integrating along the market clearing allocation $y = \mu(x)$: $u^*(x) = \int_0^x S_x(\tilde{x}, \mu(\tilde{x}))\mathrm{d}\tilde{x} + u_0$
  - This makes it clear that one needs to know the equilibrium assignment first to know the market surplus schedules
- u_0 is the constant of integration and its value depends on agent measures and the equilibrium assignment: ▸ more details
  - 0 if the measure of y is smaller than that of x
  - $S(0, \mu(0))$ if the measure of y is larger that that of x
  - any value $u_0 \in [0, S(0, \mu(0))]$ if the measures are equal
- Equilibrium $v(y)$ are given by the residual of surplus minus $u(x)$: $v^*(y) = S\left(\mu^{-1}(y), y\right) - u^*\left(\mu^{-1}(y)\right)$
- A dual problem, where x maximize their payoff by choosing a y given a schedule $v(y)$, yields the same solution

Competitive equilibrium: assortative matching

- A monotonic equilibrium is where $\mu'(x)$ is either positive or negative for all x: PAM as an allocation μ that is a strictly increasing function $(\mu'(x) > 0)$ and NAM as $\mu'(x) < 0$
- Under PAM, equilibrium allocation can be written as $\int_x^{\bar{x}} f(x)dx = \int_{\mu(x)}^{\bar{y}} g(y)dy \Leftrightarrow F(x) = G(\mu(x)) \Leftrightarrow \mu(x) = G^{-1}(F(x))$
- The properties of μ can be derived from analyzing the SOC for a global maximum, $S_{xx}(x, y) - u_{xx}(x) < 0$
- Total differentiation w.r.t x of the FOC evaluated along equilibrium allocation $y = \mu(x)$ yields $S_{xx}(x, \mu(x)) + S_{xy}(x, \mu(x))\mu'(x) = u_{xx}(x)$
- Using this identity, the SOC is therefore satisfied when $S_{xy}(x, \mu(x))\mu'(x) > 0$
- From this condition, it follows that PAM ($\mu'(x) > 0$) whenever S_{xy} is positive (S is supermodular), and NAM (i.e., $\mu'(x) < 0$) whenever f_{xy} is negative (S is submodular)

Supermodularity at two dimension

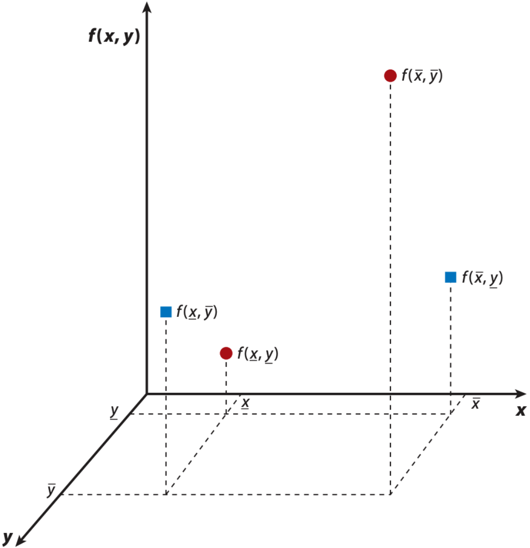


Figure 1

Supermodularity. A function $f(x, y)$ is supermodular if the sum of its value at the extremes (*red circles*) exceeds that at the intermediates (*blue squares*).

Did educational assortative mating really increase?

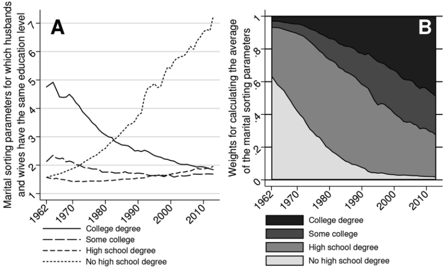


FIG. 3.—US trends in assortative mating by educational level. *A*, Time trends in the marital sorting parameters $s(e_f, e_m)$ for which the husbands and wives have the same education level. *B*, Weights used to calculate the weighted average of the marital sorting parameters along the diagonal (see fig. 4). Source: CPS (1962–2013), married couples aged 26–60.

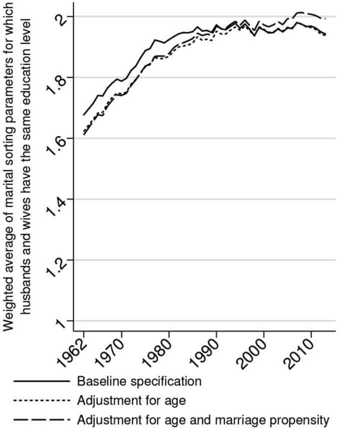


FIG. 4.—US trends in aggregate educational assortative mating. This figure displays the time trends in the weighted average of the marital sorting parameters $s(e_f, e_m)$ along the diagonal (where husbands and wives have the same education level). Source: CPS (1962–2013), married couples aged 26–60.

Assortative Mating on College Major

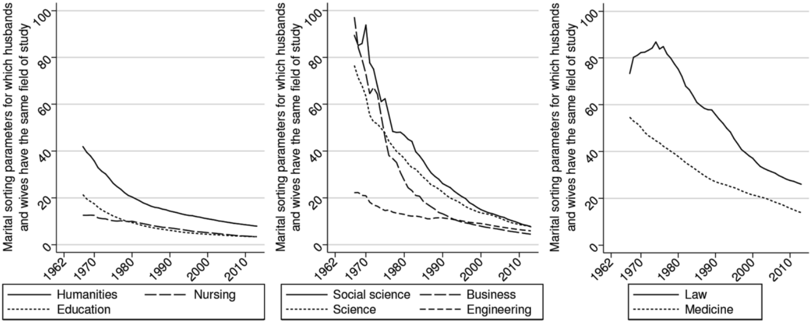
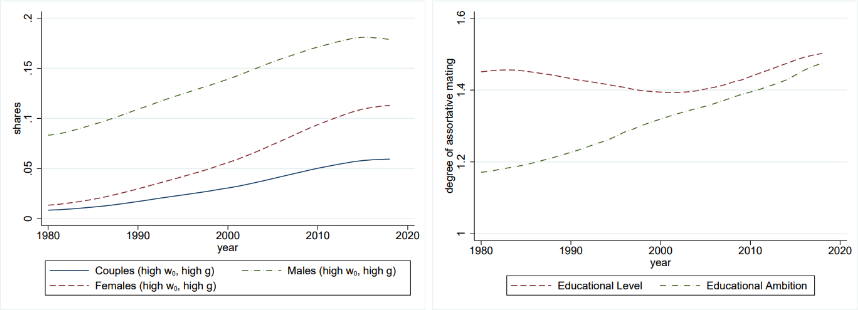


FIG. 10.—Trends in assortative mating by postsecondary field of study in Norway. This figure displays the time trends in marital sorting parameters $s(e_f, e_m)$ for which husbands and wives have the same college major. Source: Norwegian registry data (1967–2013), married couples aged 26–60.

Did educational assortative mating really increase?

Figure 2: The Measurement of Marital Sorting

(a) High-type couples and marginal distributions (b) Increasing sorting based on educational ambition



Note: Panel (a) shows shares of males and females in the top category of the educational-ambitions categorization, i.e., who graduated from educational programs associated with with high starting wages and high wage growth, along with the share of couples in which this is true for both spouses. Panel (b) shows the sorting measure $\mathcal{S}$ derived in Section 3.2 (equation 2) for educational-level types (red dashed line) and educational-ambition types (green dash-dotted line). Both categorizations use 4 types. The underlying educational-ambition categories are constructed using the k-means algorithm with standardized starting wages and standardized wage growth in the first ten years after graduation as inputs, see Section 3.1. Section 2 explains how the sample and the underlying labor market outcome (residualized log hourly wages) are constructed.

Competitive equilibrium: wage determination

- We now consider a worker(x)-firm(y) version of the competitive equilibrium studied above

- $\pi(y) = \max_{\tilde{x}} F(\tilde{x}, y) - w(\tilde{x})$ with FOC $F_x(x, y) - u'(x) = 0$; $w^*(x) = \int_0^x F_x(\tilde{x}, \mu(\tilde{x}))d\tilde{x} + w_0$

- Note that within the derived wage function, the equilibrium assignment (the labor market process in which employers choose workers) determines only relative wages, and their absolute levels are determined by the constant of integration, an arbitrary parameter
  - E.g. in this model all wages could be shifted up by one dollar and the FOC would continue to be satisfied
  - Also the marginal products of workers not defined
  - $dF = w'(x)\,dx + \pi'(\mu(x))\,\mu'(x)dx$

- The reserve prices (outside options) of workers and firms determine w_0 (and π_0) and absolute levels of wage and rents

Competitive equilibrium: wage determination

- Assume the outside options for all workers and firms are w^o and π^o, lower than which they will stay alone or match in other markets

- The absolute levels are determined by the conditions that hold for the last or marginal match

- Consider the case of marginal match $(x_m, \mu(x_m))$, the level of production will be $w^o + \pi^o$, while there are still workers with lower skill levels and firms with lower productivities, then we have $w_0 = w(x_m) = w^o$ and $\pi_0 = \pi(y_m) = \pi^o$

- Now consider the case of last match and denote the last 0:
  - If worker and firm measures are equal and $F(0,0) > w^o + \pi^o$, then $w_0 = w(0)$ can be any value in $(0, F(0,0) - w^o - \pi^o)$
  - If more workers than firms, the last matched worker get $w_0 = w(\mu^{-1}(0)) = w^o$, while $\pi_0 = \pi(0) = F(\mu^{-1}(0), 0) - w^o$
  - If less workers than firms, the last matched worker get $w_0 = w(0) = F(0, \mu(x)) - \pi^o$, while $\pi_0 = \pi(\mu(x)) = \pi^o$

Competitive equilibrium: functional form

- Suppose $F(x, y)$ takes CD form $x^{\alpha} y^{\beta}$ and skills and machine sizes are lognormally distributed with variances of logarithms σ_x^2 and σ_y^2
- Then using $w'(x)$ the wage function $w(x)$ takes the form $w(x) = Ax^{(\alpha\sigma_x + \beta\sigma_y)/\sigma_x} + C_w$ where A is a constant and C_w is the constant of integration
- This function will be concave, linear, or convex depending on whether $(\alpha\sigma_x + \beta\sigma_y)/\sigma_x$ is greater than, equal to or less than one
  - For example, if $\alpha + \beta = 1$ and if $\sigma_y > \sigma_x$ (i.e., machine sizes are more unequally distributed than skills), then $w(x)$ will be convex
  - However, only the wage function $w(x)$ will be observed, so that wages will appear to depend only on x
- $w - C_w$ will be lognormally distributed with variance of logarithms $\alpha\sigma_x + \beta\sigma_y$ (a linear combination of the inequalities in skill and machine size distributions)
- If the case of marginal match, $C_w = \frac{\beta\sigma_y w^o - \alpha\sigma_x \pi^o}{\alpha\sigma_x + \beta\sigma_y}$

Comparative Advantage and Log-Supermoduarity

- Sattinger's 1975 ecta "Comparative Advantage and the Distributions of Earnings and Abilities"
  - *"This paper constructs a model of the allocation of workers to jobs. The intention is to find the minimum requirements for the distribution of earnings to be different from the distribution of abilities. It is not necessary to depart from the assumptions of perfect competition or marginal productivity wage determination. All that is required is that there be comparative advantage in the performance of tasks by individuals."*

- To obtain PAM, Sattinger assumes $\frac{t(g_1,h_1)}{t(g_1,h_2)} < \frac{t(g_2,h_1)}{t(g_2,h_2)}$ where t is the the time that a worker g takes to perform a task h, which is exactly Ricardo's comparative advantage ($g1$ on $h1$ and $g2$ on $h2$)

- Note that this is similar as saying the productivity being $\frac{f(g_1,h_1)}{f(g_1,h_2)} > \frac{f(g_2,h_1)}{f(g_2,h_2)}$ and taking log we have log-supermodularity: $\ln f(g_1, h_1) + \ln f(g_2, h_2) > \ln f(g_2, h_1) + \ln f(g_1, h_2)$
  - Log-supermodular is even stronger than supermodular, so that you can find cases where the PAM assignment deviates from the allocation following comparative advantage

Asymmetric O-ring

- Kremer and Maskin (1996) extends the O-ring framework to explain the simultaneous increases in inequality and in segregation by skills as well as the decline in wages of low-skill workers
  - Increased segregation means that workers in the same firm saw increased correlation in wags and education or experiences (an economy shifting from General Motors to Microsoft & McDonald's)

- The key modification is that they assume a production function: $f(q, q') = qq'^2$
  - In general, we can have $f(q, q') = q^c q'^d$ where $0 < c < d$, and then redefine the unit of skill to obtain $f(q, q') = qq'^e$, $e > 1$
  - One can think of the q'-task (relatively sensitive to skill) as the "managerial" task and the q-task (relatively skill-insensitive) as the "assistant's" role
  - This production function is closely related to one used by Rosen (1981, 1982) and Lucas (1978): $f(q_1, \ldots, q_r, q_m) = q_m f\left(\sum_{i=1}^{r} q_i\right)$, $f' > 0$, $f'' < 0$ (DRS), where q_m is the skill of the manager, q_i is the skill of subordinate i, and r is the choice on the # of subordinates

Sorting/Segregation under asymmetric technology

- Suppose that there are just two skill-levels, L and H, where $L < H$. Now note that iff $H < \frac{1+\sqrt{5}}{2}L$, we will have cross-matching in equilibrium because $L^3 + H^3 < 2LH^2$
  - This tricky result—supermodularity while NAM—is due to the ”uni-sex” setting: supermodularity requries $S(x', y') + S(x, y) \geq S(x', y) + S(x, y')$ but not $\geq S(y, x') + S(x, y')$
- Imagine an economy begins with a skill-distribution in which L and H are fairly close in value, yielding cross-matching in equilibrium. Suppose now that the dispersion of skills increases, i.e. H increases and/or L decreases, eventually the economy will re-align so that there is only self-matching
- The asymmetry of the tasks in the production function works as a force in favor of deviating from self-matching (PAM) and cross-matching between workers of different skills
- With continuous skill distribution, the degree of asymmetry between tasks determines the level of deviation from self-matching