Labor Supply: Income and Substitution Effects

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Roadmap

1. Introduction
2. Some facts
3. Before the theory
4. Theory on labor-leisure choice
5. Some applications of the theory

Introduction

- We decide
  - whether to work or not
  - how many hours to work
  - how hard to work
  - when to quit a job
  - which skills to acquire
  - which occupations to enter
- How?
- What factors affect these decisions?
- (Q: how many hours do you work in a part-time job? What if now the wage doubled or tripled?)

Roadmap

Measures of labor supply

- Extensive margin: labor force participation rate
  - Labor force (LF) = employed (E) + unemployed (U)
  - Labor force participation rate = LF / working age population
- Intensive margin: working hour per worker

Labor force participation rate - Male

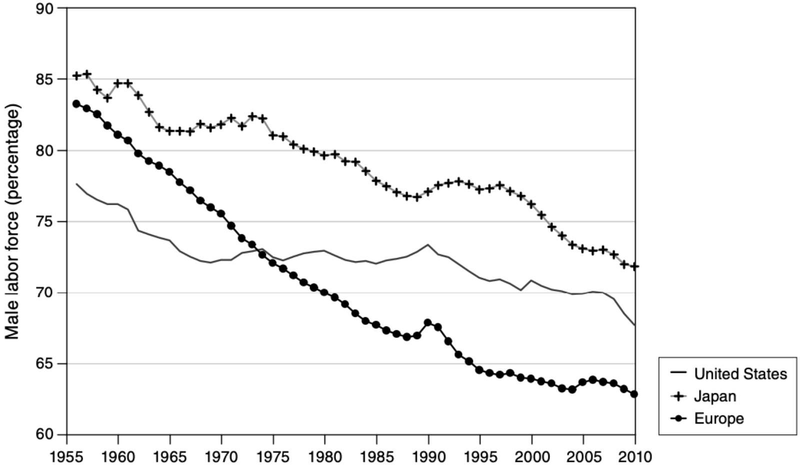


FIGURE 1.3
The evolution in civilian labor force participation rates of men in the United States, Europe, and Japan for persons 15 years of age and older, 1956–2010.

Source: OECD Annual Labor Force Statistics.

Labor force participation rate - Male by age

Table 6.2

Labor Force Participation Rates for Males in the United States, by Age, 1900–2008 (percentage)

	Age Groups					
Year	**14–19**	**16–19**	**20–24**	**25–44**	**45–64**	**Over 65**
1900	61.1		91.7	96.3	93.3	68.3
1910	56.2		91.1	96.6	93.6	58.1
1920	52.6		90.9	97.1	93.8	60.1
1930	41.1		89.9	97.5	94.1	58.3
1940	34.4		88.0	95.0	88.7	41.5
1950	39.9	63.2	82.8	92.8	87.9	41.6
1960	38.1	56.1	86.1	95.2	89.0	30.6
1970	35.8	56.1	80.9	94.4	87.3	25.0
1980		60.5	85.9	95.4	82.2	19.0
1990		55.7	84.4	94.8	80.5	16.3
2000		52.8	82.6	93.0	80.4	17.7
2008		40.1	78.7	91.9	81.4	21.5

Sources: 1900–1950: Clarence D. Long, *The Labor Force under Changing Income and Employment* (Princeton, N.J.: Princeton University Press, 1958), Table A–2.

1960: U.S. Department of Commerce, Bureau of the Census, *Census of Population, 1960: Employment Status*, Subject Reports PC(2)–6A, Table 1.

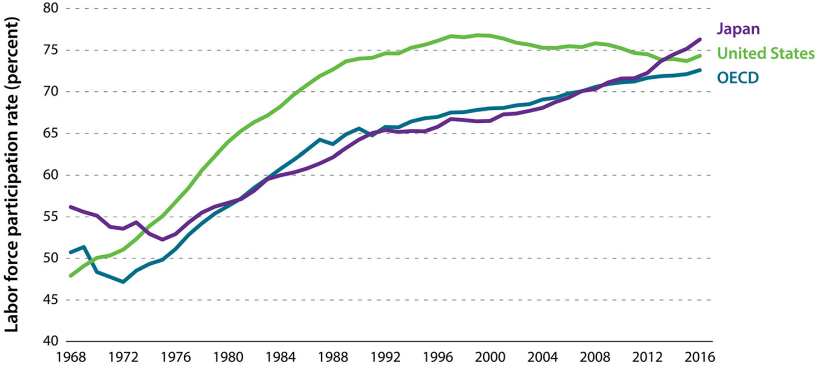
1970: U.S. Department of Commerce, Bureau of the Census, *Census of Population, 1970: Employment Status and Work Experience*, Subject Reports PC(2)–6A, Table 1.

1980–2008: U.S. Census Bureau, *2010 Statistical Abstract*, Section 12 (Table 575), http://www.census.gov/compendia/statab/2010edition.html.

Labor Force Participation Rate - Female (prime-age)

FIGURE 1.

Labor Force Participation of Prime-age Women from 1968–2016, by Country



Source: OECD Labour Force Statistics.
Note: Prime-age indicates 25–54.

Working hour per worker - Trend

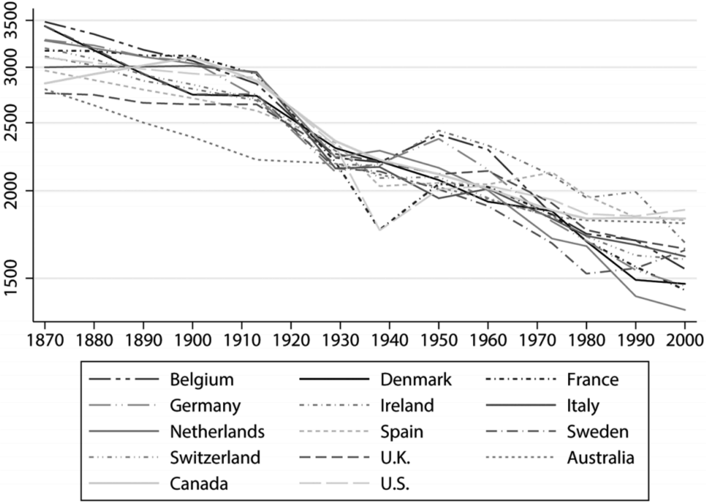


FIG. 1.—Hours worked per worker. The figure shows data for the following countries: Belgium, Denmark, France, Germany, Ireland, Italy, the Netherlands, Spain, Sweden, Switzerland, the United Kingdom, Australia, Canada, and the United States. The scale is logarithmic, which suggests that hours fall at roughly 0.57% per year. Source: Huberman and Minns (2007). Maddison (2001) shows a similar systematic decline in hours per capita. A color version of this figure is available online.

Working hour per worker - Cross-country

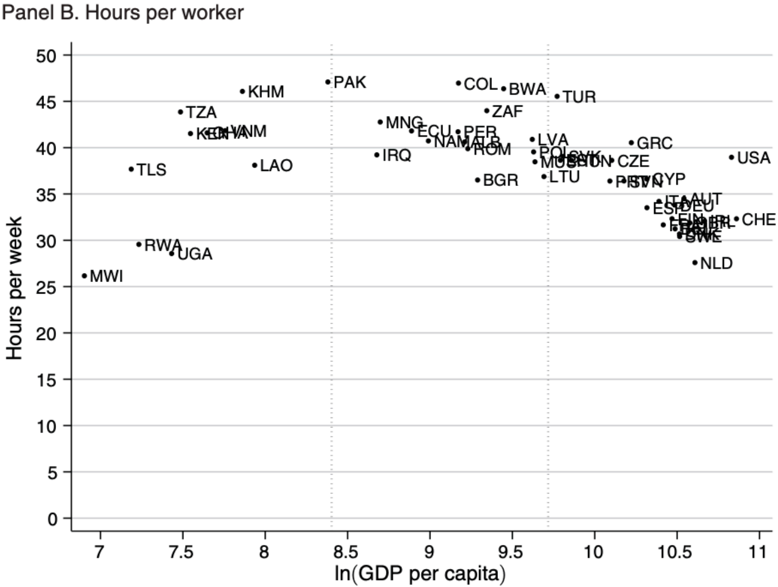


FIGURE 3. EXTENSIVE AND INTENSIVE MARGINS IN CORE COUNTRIES

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What are the potential drivers of labor supply?

- We focus on ”economics” factors
- Wage; Income; Wealth
- Leisure activities; Housework
- Taxes; Welfare policies/programs
- Economics models are games where the players act with elements and under rules, so that we can study how players behave with different elements and rules
- ”Mercenary” items are the easiest ones to be set into the game and economists typically believe they are the most powerful factors
- But even culture, belief, and identity can be modeled and studied in economics models, though they require more advanced techs

Roy Framework

- Think about a setting of either work or home production
- Two jobs:
  - Work in labor market, receive wh^m
  - Work at home and produce ph^h
- A person i works in labor market if

$$wh_i^m > ph_i^h$$

- People who are relatively more productive in the market will work
- Total labor supply depends on relative price w/p and joint distribution of human capital $F(h^m, h^h)$
- More in next week

General principles for specifying economics models

- Agents: decision-markers
  - 1 Households (preference; endowment)
  - 2 Firms (technology)
  - 3 Government (policy instruments)
- Goods: outputs and inputs
  - Output for consumption or production
  - Inputs: capital, labor (time), ...
  - Homogenous or heterogenous
- Decisions: optimizing some objectives
  - Static or dynamic decisions
- Equilibrium: how agents interact and trade goods in the markets
  - One market for one good (to clear)
  - Partial or General equilibrium
  - Competitive or imperfect competition

Can workers choose working hours?

- Don't employers set the hours of work? (e.g. Ford in 1926)
- Workers can choose part-time vs full-time
- Workers can select different industries/occupations/firms with different full-time and over-time working hours
- Workers can shirk during their working time
- Thus employer requirements eventually reflect employee preferences, esp. in the long-run
- Even cultural and political movements can be merely proximate forces with preference changes in behind
- But firms (labor demand) surely play a role in short-run (e.g. business cycle) and cross-sectional (e.g. law or IB firms) variations

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At a high level

- ▷ The neoclassical theory of labor supply (as individual choice)
- ▷ An application of consumer theory: choose between two goods (consumption and leisure)
- ▷ The tricky part: simultaneously choose consumption and ”income”
- ▷ For a more general setting of multiple goods/endowments here
- ▷ Both math and graphics would do the job here
  - ▷ Math is more generative and more accurate
  - ▷ Graphics may be more intuitive
- ▷ We abstract from any dynamics (more realistic but more complex)

Setting

- The agent has preference, i.e. a **utility function** $U(C, L)$
  - C is consumption of goods and services (with normalized $p = 1$)
  - L is leisure
  - Assume $U(\cdot, \cdot)$ is a strictly increasing and strictly concave (or strictly quasi-concave ▸ definition; intuition: decreasing marginal return)
- The agent has two endowments:
  - Disposable time T: 24 or 16 or 12 hours
  - Non-wage income Y: can be 0 or even negative (debt)
- The agent maximize utility by choosing L or working time H
  - $L + H = T$
  - Static optimization as no multiple periods and no savings
- Partial equilibrium: wage w is taken as given
  - Note the implicit assumption: w does not depend on H

Optimization

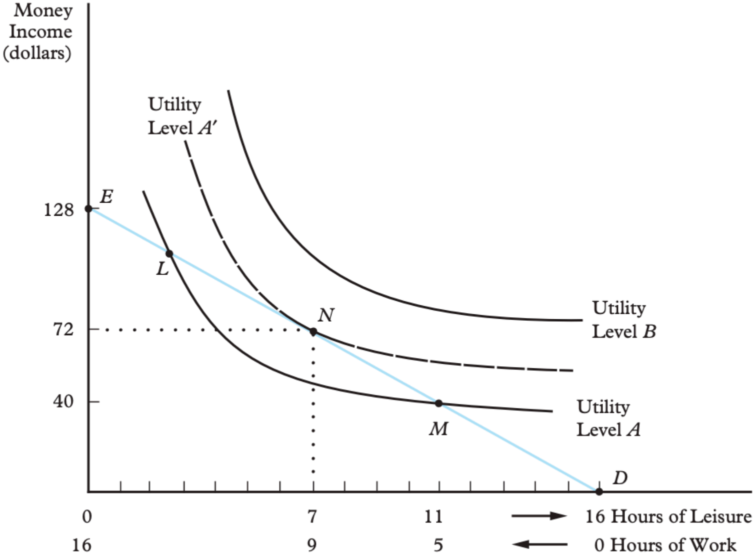
- $\max_{C,L} U(C,L)$ subject to $C = w(16 - L) + Y$
- Note the budget constraint can be also written as $16w + Y = Lw + C$
  - $16w + Y$ can be referred to as "full income"
  - The price (or opportunity cost) for L is w
  - A rise in w increases both full income and cost of leisure
- Alternatively: $\max_{C,H} V(C,H) = U(C, T - H)$ s.t. $C = wH + Y$
  - Can also set H in $V()$ as a negative term, i.e. disutility
- (Q: what are the endogenous (playable) variables and exogenous (environmental) variables in this model?)

Derviation

- $\max_{C,L} U(C,L)$ s.t. $C = w(16-L) + Y$
- Lagrangian: $\mathcal{L} = U(C,L) - \lambda\left(C - w(16-L) - Y\right)$
- Assume an interior optimum, the First Order Conditions (FOCs):
$$\mathcal{L}_C = U_C - \lambda = 0$$
$$\mathcal{L}_L = U_L - \lambda w = 0$$
$$\mathcal{L}_\lambda = C - w(16-L) - Y = 0$$
- **Tradeoff**: $U_L(C^*, L^*) = wU_C(C^*, L^*)$
  - Note U_L/U_C is the marginal rate of substitution (MRS), which equates to w, the relative price
- (Marshallian) Demand functions: $\begin{array}{l} L = L^m(w, Y) \\ C = C^m(w, Y) \end{array}$
- Lagrange multiplier: $\lambda = U_C = \lambda^m(w, Y)$ (interpreted as marginal utility or "shadow price" of income)

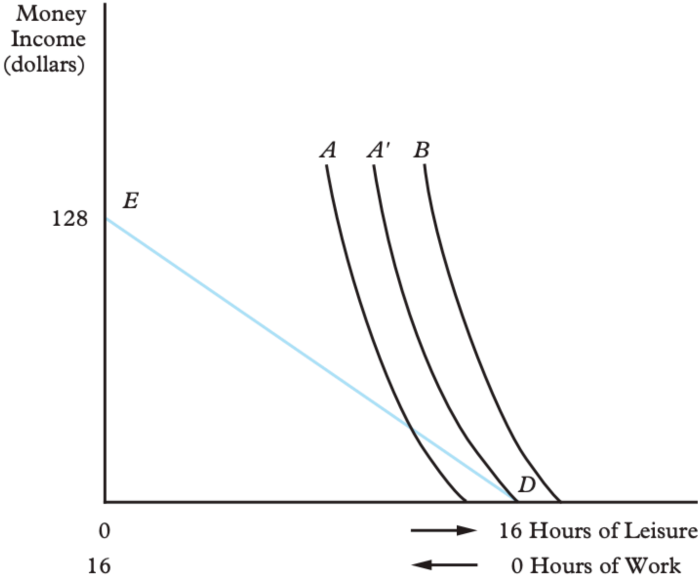
Indifference curves and budget constraint curve

Q: What if the indifference curves are concave to the origin?

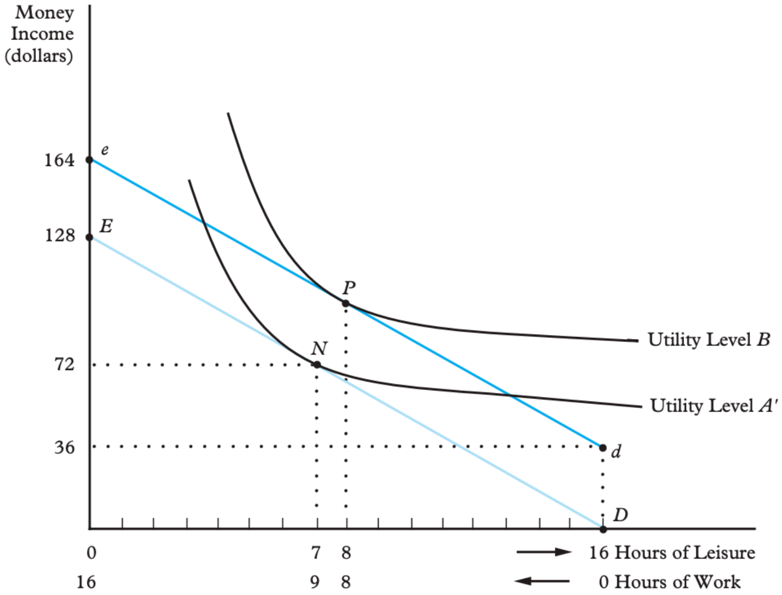


Not-work is a corner solution ($U_L > wU_C$)

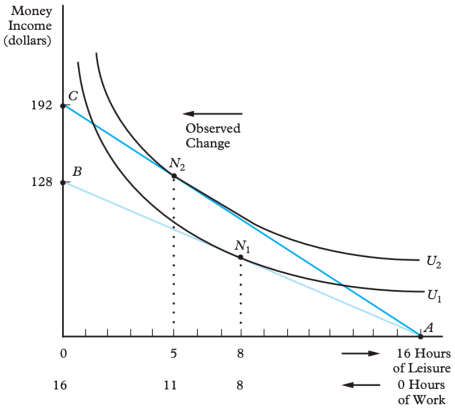
‣ reservation wage

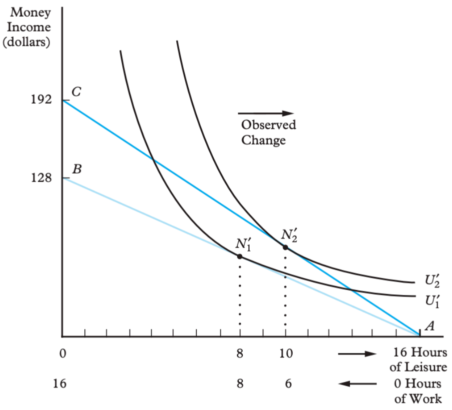


Income effect (an increase in Y)

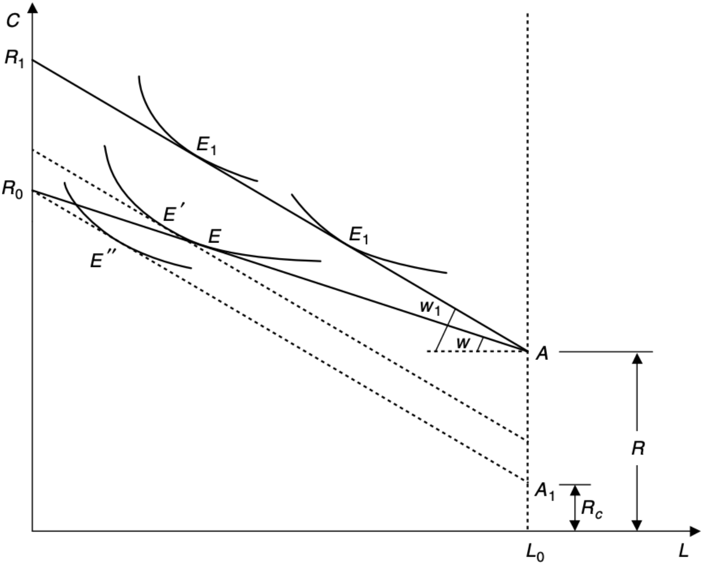


Income + Substitution effect (an increase in w)





Income + Substitution effect (an increase in w)



The dual problem

- Recall previously we have $U(C^*(w, Y), L^*(w, Y)) = V(w, Y)$, where V is the indirected utility function
- The dual problem is to minimize the expenditure to achieve some utility U: $Y(w, U) = \min_{C,L} C - w(16 - L)$ s.t. $U(C, L) \geq U$
  - Referred as "excess expenditure function"
  - Note previously we have $Y = C - w(16 - L)$, but now Y is no longer a parameter but the value of the objective function
  - (Q: how to solve this by using the graph?)
- $\mathcal{L} = C - w(16 - L) - \lambda\left(U(C, L) - U\right)$
- Hicksian (Compensated) demand functions: $\begin{array}{l} C = C^h(w, U) \\ L = L^h(w, U) \end{array}$
- Expenditure function: $Y(w, U) = C^h(w, U) - w\left(16 - L^h(w, U)\right)$

Derive Slutsky equation

- Expenditure function: $Y(w, U) = C^h(w, U) - w\left(16 - L^h(w, U)\right)$
- Sheppard's lemma: $Y_w(w, U) = -(16 - L^h(w, U)) = -H^h(w, U)$ (use Envelop theorem)
- The Hicksian and Marshallian demand functions for leisure are related to each other: $L^h(w, U) \equiv L^m(w, Y(w, U))$
- Differentiating:

$$\frac{\partial L^h}{\partial w} = \frac{\partial L^m}{\partial w} + \frac{\partial L^m}{\partial Y}\frac{\partial Y}{\partial w}$$

- Slutsky equation:

$$\frac{\partial L^m}{\partial w} = \underbrace{\frac{\partial L^h}{\partial w}}_{\text{Substitution Effect (-)}} + \underbrace{\frac{\partial L^m}{\partial Y}\left(16 - L^h\right)}_{\text{Income Effect (+)}}$$

Income/Wealth effect

- Slutsky equation:

$$\frac{\partial L^m}{\partial w} = \frac{\partial L^h}{\partial w} + \underbrace{H\frac{\partial L^m}{\partial Y}\Big|_w}_{\text{Income Effect (+)}}$$

- Income elasticity of leisure demand $\varepsilon_{L,Y} = \frac{\partial L^m}{\partial Y}\frac{Y}{L} = \frac{\partial \ln L^m}{\partial \ln Y}$
- Leisure is generally regarded as a normal good, i.e. $0 < \varepsilon_{L,Y} \leq 1$ (inferior good if $\varepsilon_{L,Y} \leq 0$; luxury good if $\varepsilon_{L,Y} > 1$)
- Quasilinear utility: $U(C, L) = C + V(L)$, where there is no income effect, i.e. $\frac{\partial L^m}{\partial Y} = 0$ ▸ an example ▸ more general case (Q: what activities?)
- Cases of observing income effect: lottery; bequest; government cash transfer (Q: when?)

Wage effect

- Slutsky equation:

$$\frac{\partial L^m}{\partial w} = \underbrace{\frac{\partial L^h}{\partial w}}_{\text{Substitution Effect (-)}} + \underbrace{\frac{\partial L^m}{\partial Y} H}_{\text{Income Effect (+)}}$$

- Elasticity form: $\underbrace{\varepsilon_{L,w}}_{\text{Uncompensated elasticity}} = \underbrace{\varepsilon^c_{L,w}}_{\text{Compensated elasticity}} + w\varepsilon_{L,Y}\frac{H}{Y}$

  (Q: does λ change when $\frac{\partial L^h}{\partial w}$? ▸ Frisch elasticity)

- Wage changes (more generally, relative price changes) due to various reasons (e.g. income tax; minimum wage) are way more likely to be observed

- The net effect depends on the relative size of two effects

- Estimated results in the microeconomics literature are rather mixed ($\varepsilon_{H,w} \in [-0.1, 0.2]$ and $\varepsilon^c_{H,w} \in [0.1, 0.3]$) and vary across different demographics (Q: how to write $\varepsilon_{H,w}$ using $\varepsilon_{L,w}$)

What do $\frac{\partial L^h}{\partial w}$ and $\frac{\partial L^m}{\partial Y}$ depend on?

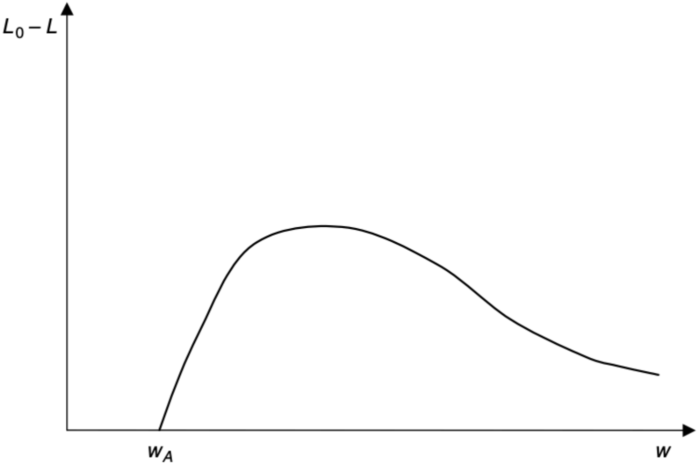
- Slutsky equation in utility terms: ‣ derivation

$$\frac{\partial L}{\partial w} = \frac{U_C - (U_{LC} - wU_{CC})(T - L)}{U_{LL} + w^2 U_{CC} - 2wU_{LC}}$$

- The denominator is the SOC of the problem and thus negative given concavity ‣ details
- Thus $\frac{\partial L}{\partial w} \propto -U_C + (U_{LC} - wU_{CC})H$
- $-U_C$ captures the substitution effect, which is proportional to the marginal utility of consumption
- $(U_{LC} - wU_{CC})H$ captures the income effect, which depends on the cross-derivative and the concavity of the utility function in consumption
  - Now you can see why for Quasi-linear utility functions the income effect is 0 ($U_{LC} = 0$; $U_{CC} = 0$)

A plausible graph of individual labor supply

Q: How would $-U_C$, U_{LC}, *and* U_{CC} *change with increase in* C*?*



Roadmap

What utility functions have no wage effect?

- Macroeconomists like to use utility functions with a form close to $u(c, l) = cv(l)$, where $v(\cdot)$ satisfies the usual conditions ▸ more general
- Recall FOC: $u_c w = u_l$
- $\Rightarrow v(l)w = cv'(l)$
- If there is no wealth, i.e. $y = 0$, then $c = wh = w(1 - l)$
- $\Rightarrow v(l)w = w(1 - l)v'(l) \Rightarrow v(l) = (1 - l)v'(l)$, i.e. l^* does not depend on w as income and substitution effects cancel (Q: show this with previous decomposition)
- If $y > 0$, $v(l) = (1 - l + y/w)v'(l)$, i.e. an increase in wage will reduce leisure as substitution effect dominates (intuition: the income effect is now smaller with $y > 0$)
  - In macro models, y and w will always grow in the same speed, so y/w is a constant and l^* will be stationary

What utility functions have declined working hour?

- MaCurdy (1981): $u(c, h) = \frac{c^{1-\sigma}-1}{1-\sigma} - \psi \frac{h^{1+1/\theta}}{1+1/\theta}$ $(\sigma, \theta \geq 0)$
- FOC: $wc^{-\sigma} = \psi h^{1/\theta}$
- $\Rightarrow h^{*} = \psi^{\frac{-1}{\sigma+1/\theta}} w^{\frac{1-\sigma}{\sigma+1/\theta}}$
- If $\sigma > 1$, h^{*} decreases with w increase, i.e. income effect dominates substitution effect (Q: show this with previous decomposition)
- If $\sigma = 1$, it return backs to previous case of perfect offsetting (Q: can we write the utility function in this case as $cv(l)$?)
- General form studied in Boppart and Krusell (2020) ▸ BK class, through which the authors support the Keynes' speculation: people will work 15-hour week in the future

What can explain declined labor supply for young men?

- Aguiar et al. (2021) suggests better recreational computing and gaming
- Agent now chooses between multiple leisure activities in addition to the work-leisure tradeoff (e.g. $\max_{c,\{\ell_1,\ldots,\ell_I\},H} U(c, v(\boldsymbol{\ell};\boldsymbol{\theta}))$ s.t. $c \leq wH$ and $\sum_{i=1}^{I} \ell_i + H \leq 1$, where $v(\boldsymbol{\ell};\boldsymbol{\theta}) = \sum_{i=1}^{I} \frac{(\theta_i \ell_i)^{1-(1/\eta_i)}}{1-(1/\eta_i)}$)
- Opportunity cost for each leisure activity is not only wage, but also the utility from choosing other activities
- They estimate this leisure demand system and find r.c.g is a "leisure luxury" specially for younger men (1% increase in leisure time associated with about a 2.5% increase in r.c.g. time)
- While the key idea is very simple, the model derivation see Appendix and estimation are nontrivial

Bring Sociology into Economics

- Thorsten Veblen proposed that consumption is motivated by a desire for social standing (along with for the enjoyment of the goods and services per se) and the the leisure class' establish the standards for the rest
- But why is it the consumption of the 'leisure class' that is emulated rather than their leisure?
  - Consumption is a more visible, i.e. costly signaling
  - Consumption and leisure can be complementary
- Bowles and Park (2005) brings this idea into the labor-leisure framework

Add emulated consumption into the framework

- Assume $u = u\left(c^o, h\right) = u\left[\left(wh - vc^r\right), h\right]$
  - $c^r \equiv w^r h^r + y$ is the consumption level of some rich reference group
  - v measure the intensity of the relevant social comparisons
- Note that vc^r here plays the same role as a negative non-wage income y, i.e. a debt
- If the utility function is Quasi-linear in h, e.g. $u = \ln c^o - \delta h$
  - Optimal solution: $h^* = 1/\delta + vc^r/w$
- $dh^*/dc^r \propto -v\left(u_{c^o h} + wu_{c^o c^o}\right)$ is positive (same is dh^*/dv)
- With many income groups each of which takes the next richest group as its reference group, an increase in consumption by the top rich generates a downward cascade of Veblen effects

Reference

Aguiar, M., M. Bils, K. K. Charles, and E. Hurst (2021). Leisure luxuries and the labor supply of young men. *Journal of Political Economy 129*(2), 337–382.

Boppart, T. and P. Krusell (2020). Labor supply in the past, present, and future: a balanced-growth perspective. *Journal of Political Economy 128*(1), 118–157.

Bowles, S. and Y. Park (2005). Emulation, inequality, and work hours: Was thorsten veblen right? *The Economic Journal 115*(507), F397–F412.

Appendix

Concavity and Quasi-concavity

- Definition. We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is concave if, for any $x, y \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$, we have: $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$.
- Definition. We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-concave if, for any $x, y \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$
- Note $\lambda f(x) + (1 - \lambda)f(y) \geq \min\{f(x), f(y)\}$, so quasi-concavity is a weaker condition than concavity
- Strictly concave or quasi-concave means replacing $\geq$ with $>$
- Example of strictly concave function: $U(x, y) = x^\alpha y^{1-\alpha}$ (Cobb-Douglas)
- Example of concave function: $U(x, y) = ax + by$ (Linear)
- Example of quasi-concave but not concave function: $U(x, y) = \min(ax, by)$ (Leontief)

Concavity and Second Derivative

- Assume a univariate function $f : \mathbb{R} \to \mathbb{R}$ has $f''(x) \leq 0$ for all $x \in \mathbb{R}$
- Recall Taylor's Expansion:
  $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(\xi)(x - x_0)^2$, where ξ is some point between c and x
- Since $f''(x) \leq 0$, the last term is non-positive
- Let $x_0 = \lambda x_1 + (1 - \lambda)x_2$ and take $x = x_1$, we have
  $f\left(x_1\right) \leq f\left(x_0\right) + f'\left(x_0\right)\left(\left(1 - \lambda\right)\left(x_1 - x_2\right)\right)$
- Simiarly, taking $x = x_2$, $f\left(x_2\right) \leq f\left(x_0\right) + f'\left(x_0\right)\left(\lambda\left(x_2 - x_1\right)\right)$
- Multiplying $f\left(x_1\right)$ by λ and $f\left(x_2\right)$ by $1 - \lambda$ and adding, we have
  $\lambda f\left(x_1\right) + (1 - \lambda)f\left(x_2\right) \leq f(x_0) = f(\lambda x_1 + (1 - \lambda)x_2)$
- For multivariate functions, the requirement is more complex: we need the Hessian matrix H to be negative semi-definite

General economy

- Consumer comes to the market with initial endowments of $n+1$ goods $\{x_0^0, x_1^0, \ldots, x_n^0\}$
- Market sets prices of $p_0, p_1, \ldots, p_n$ for these goods
- Consumer trade in the markets and maximize utility by buying and selling goods: $\max U(x_0, x_1, \ldots, x_n)$ s.t. $\sum_{i=0}^{n} p_i x_i^0 = \sum_{i=0}^{n} p_i x_i$
- $\mathcal{L} = U(x_0, \ldots, x_n) + \lambda\left(\sum p_i x_i^0 - \sum p_i x_i\right)$

$$\mathcal{L}_0 = U_0 - \lambda p_0 = 0$$
$$\mathcal{L}_1 = U_1 - \lambda p_1 = 0$$

- $\vdots$

$$\mathcal{L}_n = U_n - \lambda p_n = 0$$
$$\mathcal{L}_\lambda = \sum p_i x_i^0 - \sum p_i x_i = 0$$

- $x_i = x_i^m\left(p_0, \ldots, p_n, x_0^0, \ldots, x_n^0\right) \quad i = 0, \ldots, n$

General economy

- Set the good x_0 as numeraire (i.e., $p_0 = 1$) and the amount of x_0^0 as the excess expenditure (money income)
- The dual problem: $e = \min \sum_{i=0}^{n} p_i x_i - \sum_{i=1}^{n} p_i x_i^0$
  s.t. $U\left(x_0, \ldots, x_n\right) = U$
- Excess expenditure function (indirect "endowment function"):
  $e\left(p_1, \ldots, p_n, x_1^0, \ldots, x_n^0, U\right) = \sum_{i=0}^{n} p_i x_i^h - \sum_{i=1}^{n} p_i x_i^0$
- $\frac{\partial e}{\partial p_i} = x_i^h - x_i^0$
- $x_i^h\left(p_1, \ldots, p_n, U\right) = x_i^m\left(p_1, \ldots, p_n, e, x_1^0, \ldots, x_n^0\right)$
- $\frac{\partial x_i^h}{\partial p_j} = \frac{\partial x_i^m}{\partial p_j} + \frac{\partial x_i^m}{\partial e}\frac{\partial e}{\partial p_j}$
- $\frac{\partial x_i^m}{\partial p_j} = \frac{\partial x_i^h}{\partial p_j} + \frac{\partial x_i^m}{\partial e}\left(x_j^0 - x_j^h\right)$

Reservation wage

- Recall $wU_C - U_L < 0$ for not-work agents
- We can define the reservation wage w^* by $w^* = \frac{U_L(Y,T)}{U_X(Y,T)}$, i.e. the wage that is just high enough to induce the agent to supply a tiny unit of labor
- Examples: vendors in sports stadium; construction workers
- Alternatively, since now $C = Y$, reducing Y can increase U_X and induce labor supply

Derive Slutsky equation using utility function

- Total differentiating $\frac{U_L}{U_C} = w$ with respect to w
- $\frac{U_C \frac{\partial U_L}{\partial w} - U_L \frac{\partial U_C}{\partial w}}{U_C^2} = 1 \Rightarrow \frac{\partial U_L}{\partial w} - \frac{U_L}{U_C}\frac{\partial U_C}{\partial w} = U_C \Rightarrow \frac{\partial U_L}{\partial w} - w\frac{\partial U_C}{\partial w} = U_C$
- $\Rightarrow U_{LL}\frac{\partial L}{\partial w} + U_{LC}\frac{\partial C}{\partial w} - w(U_{CC}\frac{\partial C}{\partial w} + U_{LC}\frac{\partial L}{\partial w}) = U_C$
- From $C = (T-L)w + Y \Rightarrow \frac{\partial C}{\partial w} = T - L - w\frac{\partial L}{\partial w}$
- $\Rightarrow U_{LL}\frac{\partial L}{\partial w} + (U_{LC} - wU_{CC})(T-L) + w^2 U_{CC}\frac{\partial L}{\partial w} - 2wU_{LC}\frac{\partial L}{\partial w} = U_C$
- $\Rightarrow \frac{\partial L}{\partial w} = \frac{U_C - (U_{LC} - wU_{CC})(T-L)}{U_{LL} + w^2 U_{CC} - 2wU_{LC}}$
- Note that you can totally differentiating w.r.t. Y to get $\frac{\partial L}{\partial Y}$, which directly gives you the formula of income effect! (In fact, the more general way to do all the derivations is to total differentiate FOCs w.r.t w and Y in the matrix form and then to solve the system)

Second Order Condition

- As we are dealing with constrained optimization, we examine how the objective function changes according to a vector of budget-neutral variations: $(dC, dL) = (-w, 1)dL$ (because we need $dC = -wdL$ for budget not changing)
- The second-order effect of such a variation is
$$(-w, 1)\begin{bmatrix} U_{CC} & U_{CL} \\ U_{LC} & U_{LL} \end{bmatrix}\begin{bmatrix} -w \\ 1 \end{bmatrix} = w^2 U_{CC} - 2wU_{CL} + U_{LL} < 0$$
  - $\begin{bmatrix} U_{CC} & U_{CL} \\ U_{LC} & U_{LL} \end{bmatrix}$ is the Hessian matrix of the utility function
  - Quadratic form ($v'Qv$) here tells about the steepness or curvature of the specific path following our budget-neutral variations
  - We can prove this inequality holds with strictly quasi-concave (s.q.c) utility function (see next slide)
  - In fact, the concavity assumed already ensures the Hessian matrix to be negative semi-definite, i.e. the quadratic form to be negative for all non-zero vectors v
  - That's why FOCs are both necessary and sufficient to characterize an interior "preference maximal" with s.q.c!

S.Q.C and SOC

- Assume $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly quasi-concave
- Let $x^* \in \mathbb{R}^n$ be a critical point where the FOCs are satisfied
- Let t be any non-zero vector such that $p \cdot t = 0$, i.e., t is tangent to the budget constraint ($p \cdot x = I$)
- Pick two points, $x_1 = x^* + \epsilon t$ and $x_2 = x^* - \epsilon t$
- Given s.q.c, for any $\alpha \in (0, 1)$,
$$u\left(\alpha\left(x^* + \epsilon t\right) + (1 - \alpha)\left(x^* - \epsilon t\right)\right) > \min\left\{u\left(x^* + \epsilon t\right), u\left(x^* - \epsilon t\right)\right\}$$
$$u\left(x^*\right) > \min\left\{u\left(x^* + \epsilon t\right), u\left(x^* - \epsilon t\right)\right\}$$
- Using the Taylor series expansion ($\epsilon \rightarrow 0$):
$$u\left(x^* \pm \epsilon t\right) \approx u\left(x^*\right) \pm \epsilon t' Du\left(x^*\right) + \frac{1}{2}\epsilon^2 t' D^2 u\left(x^*\right) t$$
- Note that the second linear term is 0 under FOCs, and thus the quadratic term, $t' D^2 u\left(x^*\right) t$, must be negative for s.q.c to hold!

Example of Quasi-linear utility

- $U(C, L) = C + \frac{L^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}$
- The optimality condition: $L^{\frac{1}{\varepsilon}} = w$
- Note that the optimal choice of L is not a function of income (or more accurately, not a function of consumption c), i.e. $\varepsilon^{u}_{L,w} = \varepsilon^{c}_{L,w}$
- $\varepsilon^{u}_{L,w} = \partial \log L / \partial \log w = \varepsilon$
- Thus wage elasticity is a constant, i.e. this utility function has a constant elasticity of labor supply (purely through the substitution effect)

Derive Frisch (λ-constant) elasticity

- FOCs: $$U_C = \lambda$$ $$U_L = \lambda w$$
- Define Frisch demand $L^f(w, \lambda)$ implicitly by $U_L(L^f(w, \lambda)) = \lambda w$ (same for C^f with $p = 1$)
- Totally differentiating while holding a constant λ:

$$\begin{bmatrix} U_{CC} & U_{CL} \\ U_{LC} & U_{LL} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial C^f}{\partial w^f} \\ \frac{\partial L^f}{\partial w} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial C^f}{\partial w} \\ \frac{\partial L^f}{\partial w} \end{bmatrix} = \begin{bmatrix} U_{CC} & U_{CL} \\ U_{LC} & U_{LL} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$$

- $$= \frac{1}{U_{CC}U_{LL} - U_{CL}^2} \begin{bmatrix} U_{LL} & -U_{CL} \\ -U_{LC} & U_{CC} \end{bmatrix} \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\lambda U_{CL}}{U_{CC}U_{LL} - U_{CL}^2} \\ \frac{\lambda U_{CC}}{U_{CC}U_{LL} - U_{CL}^2} \end{bmatrix} = \begin{bmatrix} \frac{U_C U_{CL}}{U_{CC}U_{LL} - U_{CL}^2} \\ \frac{U_C U_{CC}}{U_{CC}U_{LL} - U_{CL}^2} \end{bmatrix}$$

A comparison among elasticities

- $\varepsilon^{u}_{L,w} \geq \varepsilon^{c}_{L,w}$ since the income effect is positive
- $$\begin{aligned} \frac{1}{\varepsilon^{c}_{L,w}} - \frac{1}{\varepsilon^{f}_{L,w}} &= \frac{U_{LL} + w^2 U_{CC} - 2wU_{LC}}{U_C} - \frac{U_{CC}U_{LL} - U_{CL}^2}{U_C U_{CC}} \\ &= \frac{1}{U_C}\left(w^2 U_{CC} - 2wU_{LC} + \frac{U_{CL}^2}{U_{CC}}\right) \end{aligned}$$
- The definition of λ-constant elasticity implies that $U_{LL} \leq \frac{U_{CL}^2}{U_{CC}}$.
- $$\begin{aligned} w^2 U_{CC} - 2wU_{LC} + \frac{U_{CL}^2}{U_{CC}} &\leq w^2 U_{CC} - 2wU_{LC} + U_{LL} \\ &= SOC \\ &\leq 0 \end{aligned}$$
- Thus $\varepsilon^{f}_{l,w} \leq \varepsilon^{c}_{l,w}$ (Q: for what utility function does equality hold?)

KPR class of utility functions

- King, Plosser, Rebelo (1988) show that balanced growth with constant hours worked is obtained only if the period utility function is $u(c,h) = \begin{cases} \frac{(c \cdot v(h))^{1-\sigma}-1}{1-\sigma} \text{ if } \sigma \neq 1 \\ \log(c) + \log v(h) \text{ if } \sigma = 1 \end{cases}$
- Note that this is just putting $cv(h)$ into a CRRA utility function
- Two special cases
- 1) $u(c,h) = \begin{cases} \frac{(c(1-h)^{\kappa})^{1-\sigma}}{1-\sigma} \text{ if } \sigma \neq 1 \\ \log(c) + \kappa \log(1-h) \text{ if } \sigma = 1 \end{cases}$
  - Cobb-Douglas, i.e. elasticity of substitution between c and l is 1
- 2) $u(c,h) = \log(c) - \psi \frac{h^{1+1/\theta}}{1+1/\theta}$ (especially common!)
  - Constant Frisch elasticity ($\varepsilon_t^f = \frac{u_h}{h_t\left[u_{hh} - \frac{u_{hc}^2}{u_{cc}}\right]}$) when $\theta > 0$, which will be akin to the expression for IES (i.e. inverse of risk aversion)

BK class of utility functions

- Boppart & Krusell (2020) extend the KPR class to allow working hour change at a constant rate:

$$u(c,h) = \begin{cases} \frac{\left(c \cdot v\left(hc^{\nu/(1-\nu)}\right)\right)^{1-\sigma} - 1}{1-\sigma} \text{ if } \sigma \neq 1 \\ \log(c) + \log(v(hc^{\nu/(1-\nu)})) \text{ if } \sigma = 1 \end{cases}$$

- For $\nu > 0$, $c^{\frac{\nu}{1-\nu}}$ captures the stronger income effect: an added "penalty" to working (since v is decreasing)

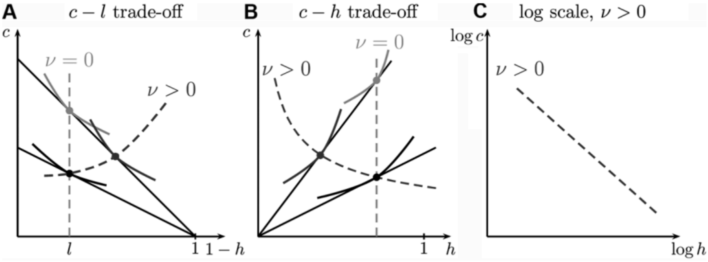


FIG. 5.—Consumption-leisure trade-off. The figure panels abstract from unearned income. A color version of this figure is available online.

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GHH preference

- Another popular utility specification for macroeconomist is the GHH class (Greenwood, Hercowitz, Hoffman 1998):

$$u(c,h) = \begin{cases} \frac{\left(c - \psi \frac{h^{1+1/\theta}}{1+1/\theta}\right)^{1-\sigma}}{1-\sigma} & \text{if } \sigma \neq 1 \\ \log\left(c - \psi \frac{h^{1+1/\theta}}{1+1/\theta}\right) & \text{if } \sigma = 1 \end{cases}$$

  - Note that this form looks similar to 2nd special case in KPR class
  - Like KPR, GHH also features non-separability between consumption and leisure/labor (when $\sigma \neq 1$)
  - Unlike KPR, GHH preferences are not consistent with balanced growth because it eliminates the income/wealth effect on labor supply

- FOC: $\psi h^{\theta} = w$
  - Thus labor is only a function of the wage (not of consumption)

ABCH2021: Preferences

- Assume $U(c, v(\boldsymbol{h}; \boldsymbol{\theta}, \boldsymbol{\xi}))$ with weak separability
- $v(\boldsymbol{h}_k; \boldsymbol{\theta}, \boldsymbol{\xi}_k) = \sum_{i=1}^{I} \frac{(\theta_i \xi_{ik} h_{ik})^{1-(1/\eta_i)}}{1-(1/\eta_i)}$ (k index individuals)
  - $\boldsymbol{h} = \{h_1, \ldots, h_I\}$ is time spent on I leisure activities
  - $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_I\}$ is a vector of technology shifters
  - $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_I)$ are idiosyncratic preferences over activities
  - $\eta_i > 0$ governs the diminishing returns
- $\max_{c, \{h\}, N} \{U(c, v(\boldsymbol{h}; \boldsymbol{\theta}, \boldsymbol{\xi})) + \lambda(wN - c)\}$,
  s.t. $\sum_{i=1}^{I} h_i + N \leq 1, \quad N \in \mathcal{N}$
- FOC: $U_c = \lambda$; $U_v v_i = \omega \ \forall i$, where $v_i = \partial v / \partial h_i$
- Denote $\hat{\omega} \equiv \omega / U_v$ as normalized (shadow) price of time, which is sufficient to determine the allocation of activities
- The analysis is done for a fixed λ (abstract from income effect)

ABCH2021: Leisure Engel Curves

- Subproblem: $\mathrm{v}(H;\boldsymbol{\theta},\boldsymbol{\xi}) \equiv \max_{\{h_i\}} v\left(h_1,\ldots,h_I;\boldsymbol{\theta},\boldsymbol{\xi}\right)$ s.t. $\sum_i h_i \leq H$
- $\mathrm{v}_H(H;\theta,\xi) = \hat{\omega}$
- FOC: $h_i = (\theta_i\xi_i)^{\eta_i-1}\,\hat{\omega}^{-\eta_i}$
  - $\partial \ln h_i / \partial \ln \hat{\omega} = -\eta_i$
  - Note if θ_i or ξ_i increase or decrease h_i will depend on whether $\eta_i \gtrless 1$ (Q: can you think a case $\eta_i < 1$?)
- $H = \sum_i h_i = \sum_i (\theta_i\xi_i)^{\eta_i-1}\,\hat{\omega}^{-\eta_i}$
- Differentiating w.r.t H : $\frac{\partial \ln \mathrm{v}_H}{\partial \ln H} = \frac{-1}{\sum_i s_i\eta_i} = \frac{-1}{\bar{\eta}}$, where $s_i = h_i/H$
- Similarly: $\frac{\partial \ln \mathrm{v}_H}{\partial\theta_i} = \frac{\partial \ln \mathrm{v}_H}{\partial\xi_i} = \frac{s_i(\eta_i-1)}{\bar{\eta}}$ (Q: typo?)
- "Leisure Engel curve": $\beta_i \equiv \frac{\partial \ln h_i}{\partial \ln H} = \frac{\partial \ln h_i}{\partial \ln \mathrm{v}_H}\frac{\partial \ln \mathrm{v}_H}{\partial \ln H} = \frac{\eta_i}{\bar{\eta}}$

ABCH2021: Inferring Technological Progress

- Let $j \neq i$ be a "reference activity" with no changes in θ_j (e.g. sleeping)
- From FOC: $\frac{\ln h_i}{\eta_i} - \frac{\ln h_j}{\eta_j} = \left(\frac{\eta_i - 1}{\eta_i}\right) \ln \theta_i \xi_i - \left(\frac{\eta_j - 1}{\eta_j}\right) \ln \theta_j \xi_j$
- Difference over time (with invariant ξs):
$\frac{\Delta \ln h_i}{\eta_i} - \frac{\Delta \ln h_j}{\eta_j} = \left(\frac{\eta_i - 1}{\eta_i}\right) \Delta \ln \theta_i$
- $\Delta \ln \theta_i = \frac{1}{\beta_i \bar{\eta} - 1} \left(\Delta \ln h_i - \frac{\beta_i}{\beta_j} \Delta \ln h_j\right)$ (use $\eta_i = \beta_i \bar{\eta}$)
- With estimated ηs and observed hs, we can identify $\Delta \ln \theta_i$

ABCH2021: Technology and Shadow Value of Time

- $U(c, v(\boldsymbol{h}; \boldsymbol{\theta}, \boldsymbol{\xi})) = U(c, \mathrm{v}(H; \boldsymbol{\theta}, \boldsymbol{\xi}))$; $c = C(\lambda, \mathrm{v}(H; \theta, \xi))$ given by inverting $U_c = \lambda$
- $U_v \mathrm{v}_H = \omega$ (can thus write $\omega(H; \lambda, \theta, \xi)$)
- $\Rightarrow -\left(\frac{U_{vv} - U_{cv}^2 / U_{cc}}{U_v}\right) \mathrm{v}_H H = \frac{1}{\epsilon} - \frac{1}{\bar{\eta}}$, where $\epsilon \equiv -\frac{\partial \ln H}{\partial \ln \omega}$

$$\frac{\partial \ln \omega}{\partial \ln \theta_i} = \left(\frac{U_{vv} - U_{cv}^2 / U_{cc}}{U_v}\right) \mathrm{v}_{\theta_i} \theta_i + \frac{\partial \ln \mathrm{v}_H}{\partial \ln \theta_i}$$

- $\Rightarrow$

$$= \left(\frac{U_{vv} - U_{cv}^2 / U_{cc}}{U_v}\right) s_i \mathrm{v}_H H + \frac{s_i (\eta_i - 1)}{\bar{\eta}}$$

$$= \frac{s_i (\beta_i \epsilon - 1)}{\epsilon}$$

- $\Rightarrow \Delta \ln \omega \approx \frac{\partial \ln \omega}{\partial \ln \theta_i} \Delta \ln \theta_i = \frac{s_i}{\epsilon} \left[\frac{\beta_i \epsilon - 1}{\beta_i \bar{\eta} - 1}\right] \left(\Delta \ln h_i - \frac{\beta_i}{\beta_j} \Delta \ln h_j\right)$

ABCH2021: Response of Labor Supply to Technology

- FOC for N: $U_v \mathbf{v}_H = \omega = \lambda w$
- Recall we hold λ constant, thus same as $\ln \omega(H; \theta, \xi) - \ln w$
- Differentiating: $-\frac{\partial \ln H}{\partial \ln w} = -\frac{\partial \ln H}{\partial \ln \omega} = \epsilon$ (i.e. Frisch elasticity of leisure)
- $\Rightarrow \frac{d \ln H}{d \ln \theta_i} = -\frac{\partial \ln \omega / \partial \ln \theta_i}{\partial \ln \omega / \partial \ln H} = \epsilon \frac{\partial \ln \omega}{\partial \ln \theta_i} = s_i \left(\beta_i \epsilon - 1\right)$
- $\Rightarrow \frac{d \ln N}{d \ln \theta_i} = -\varphi_{\ln} \frac{\partial \ln \omega}{\partial \ln \theta_i} = -\left(\frac{\varphi_{\ln}}{\epsilon}\right) s_i \left(\beta_i \epsilon - 1\right)$ where $\varphi_{\ln} \equiv -(H/1 - H)\epsilon$ (intensive-margin Frisch elasticity)

ABCH2021: Response of Labor Supply (Extensive)

- Assume $\mathcal{N} = \{0, \bar{n}\}$ and $U_{cv} = 0$ (additive separability)
- An individual chooses employment if $\lambda w \bar{n} \geq \Delta U$, where $\Delta U \equiv U(c, \mathrm{v}(1, \theta, \xi) - U(c, \mathrm{v}(1 - \bar{n}, \theta, \xi))$ (leisure cost)
- We can define reservation wage: $w^{\mathrm{R}} = \frac{\Delta U}{\lambda \bar{n}}$
- Taking a second-order approximation of ΔU around $H = 1 - \bar{n}$: $\Delta U \approx U_v \mathbf{v}_H \bar{n} + \dfrac{1}{2}\left(U_{vv}\mathbf{v}_H^2 + U_v \mathbf{v}_{HH}\right)\bar{n}^2 = \omega\left(1 - \dfrac{1}{2\epsilon}\dfrac{\bar{n}}{1 - \bar{n}}\right)\bar{n}$
- Combining: $\ln w^{\mathrm{R}} = \ln \omega + \ln\left(1 - \frac{1}{2\epsilon}\frac{\bar{n}}{1-\bar{n}}\right) - \ln \lambda$
- With a common market wage, fraction employed is $E = \Pr\left(\ln w^{\mathrm{R}} \leq \ln w\right) = F(\ln w)$
- $\varphi_{\mathrm{Ex}} \equiv d \ln E / d \ln w = f(\ln w) / F(\ln w)$ (extensive-margin Frisch elasticity)
- $\frac{d \ln E}{d \ln \theta_i} = -\varphi_{\mathrm{Ex}} \frac{\partial w^{\mathrm{R}}}{\partial \ln \theta_i} = -\left(\frac{\varphi_{\mathrm{Ex}}}{\epsilon}\right) s_i \left(\beta_i \epsilon - 1\right)$ (assume $\frac{\partial \epsilon}{\partial \theta_i} = 0$)